Probable Factors

Teacher Notes

7 8 **9** **10** **11** 12

TI-Nspire CAS | Investigation | Student | 120min

Problem Statement:

What is the probability that a randomly generated quadratic can be factorised? This investigation looks at a significantly reduced set of quadratics using dice to generate the coefficients. Two dice are rolled; the numbers appearing uppermost are added together and form the x coefficient (b) in the quadratic, $y = x^2 + bx + c$. Another two dice are rolled; the numbers appearing uppermost multiply together and form the constant (c). The coefficient of x^2 is 1. What is the probability this quadratic can be factorised:

- Over the rational number field?
- Over the real number field?

Number & Algebra – Year 9: Patterns and Algebra

Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate (ACMNA213)

ScOT: Binomials,

Number & Algebra – Year 10: Patterns and Algebra

Expand binomial products and factorise monic quadratic expressions using a variety of strategies (ACMNA233)

Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate (ACMNA239)

ScOT: Quadratic Equations, Distributivity, Monomials,

Statistics & Probability – Year 10: Chance

Describe the results of two- and three-step chance experiments, both with and without replacements, assign probabilities to outcomes and determine probabilities of events. Investigate the concept of independence (ACMSP246)

ScOT: Probability, Experiments

Equipment

For this activity you will need:

- Four regular dice
- TI-nspire CAS

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Before commencing the digital simulation, it is important for students to generate a number of quadratics by hand using regular dice. This helps students understand what the simulation is doing and establishes a clear focus for the investigation.

Teacher Demonstration:

Roll two dice and write the results on the board for students to see, including the first part of the quadratic.

Example: (3, 5) Coefficient: $3 + 5 = 8$ Quadratic: $x^2 + 8x +$

Roll the second pair of dice, write the results on the board and complete the quadratic:

Example: (6, 2) Constant: $6 \times 2 = 12$ Quadratic: $x^2 + 8x + 12$

Does this quadratic factorise?

Example: $x^2 + 8x + 12 = (x + 6)(x + 2)$

This example represents 'success' as the randomly produced quadratic factorised.

Repeat this process, rolling new numbers for the coefficient and the constant.

Example: (4, 1) Coefficient: $4 + 1 = 5$ Quadratic: $x^2 + 5x +$

Roll the second pair of dice, write the results on the board and complete the quadratic:

Example: (5, 2) Constant: $5 \times 2 = 10$ Quadratic: $x^2 + 5x + 10$

Does this quadratic factorise?

Example: $x^2 + 5x + 10 =$ *Does not factorise*

This example represents 'failure' as the randomly produced quadratic did not factorise.

Students should now try one for themselves. Ask students to volunteer their randomly generated quadratics and write them on the board. How many of the quadratics factorise? What is the probability that quadratics produced using this process will factorise over the rational number field? (This is the focus for the investigation.)

The quantity of calculations required to determine the theoretical probability is quite significant. It is therefore desirable to establish an estimate using a simulation. The simulation can be achieved using the calculator and the instructions provided. It is possible to produce the document and send it to students rather than have each student create the simulation document independently. This choice is left to the individual and depends on how much time is to be devoted to learning how to use the calculator as an exploratory tool and how much is to focus specifically on identifying and answer from the simulation.

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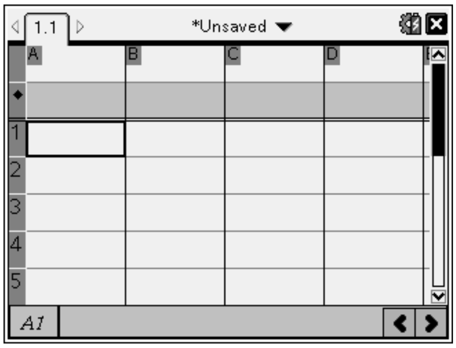
Introduction – Setting up the simulation

Start a new document and insert a spreadsheet. The spreadsheet will be used to generate the random integers (whole numbers) in place of the dice.

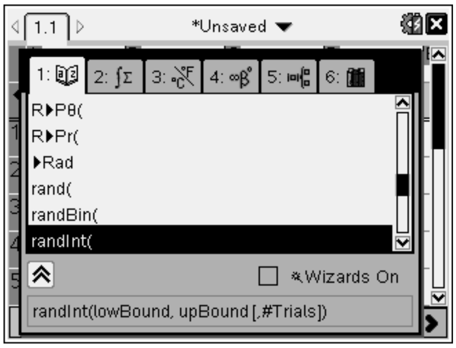


In cell A1 type the formula: =randInt(1,6)

This produces a random integer between 1 and 6.

Note:

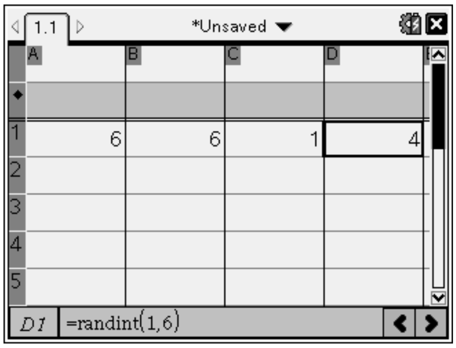
In the spreadsheet application the fastest way to access the **randInt(** command is through the catalogue.



The formula in cell A1 can be copied into cells B1, C1 and D1.

Click on cell A1, press **CTRL + C** to copy, move to cell B1 and press **CTRL + V** to paste.

Repeat this process for cells C1 and D1.



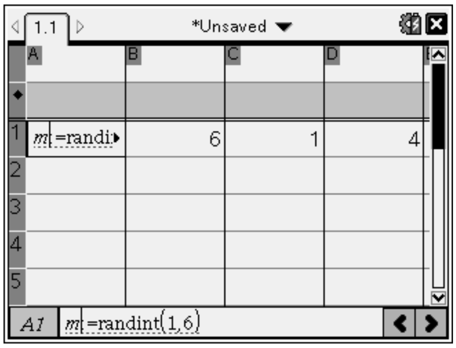
The simulated dice rolls need to be stored. Select cell A1 and press **CTRL** + **VAR** and store the dice roll in ‘m’.

Repeat this for cell B1, store as ‘n’; cell C1 store as ‘p’ and cell D1 store as ‘q’.

Note:

The simulated dice rolls become bold once they have been stored.

Press **CTRL + R** to simulate dice rolls and observe how each cell contains a new outcome each time CTRL + R is pressed.



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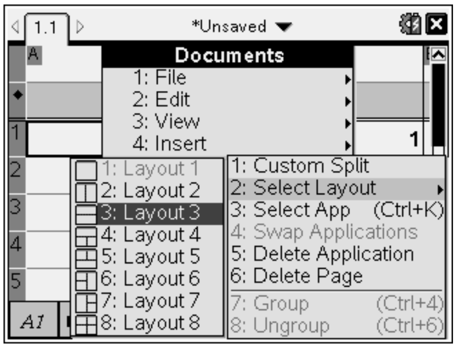
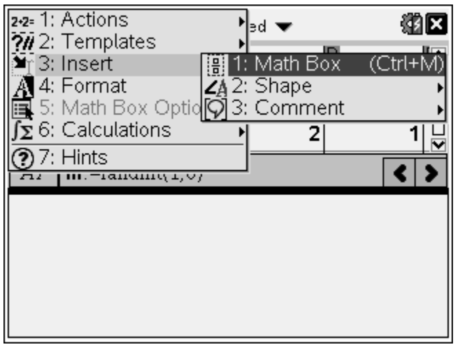
The next step is to split the current page into two applications. Use the document key (or CTRL + Home on clickpad), select **Page Layout** followed by **Select Layout** and split the screen into two horizontal regions.



Use the mouse or **CTRL + Tab** to switch between applications. When the new region is active use the menu to insert a 'Notes' application.

On the notes page use the **menu** to **insert** a **Maths Box**.

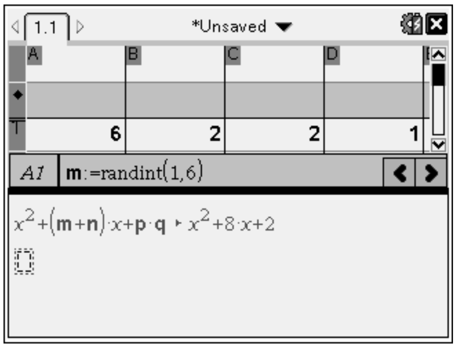


In the Maths box type: $x^2+(m+n)\cdot x+p\cdot q$

Note:

A multiplication sign must be used between the parenthesis and the x, similarly between the p and q to ensure these operations are multiplied.

Notice the defined variables m, n, p and q appear bold. Furthermore the expression is automatically simplified as the output for the maths box. Can the expression be factorised?

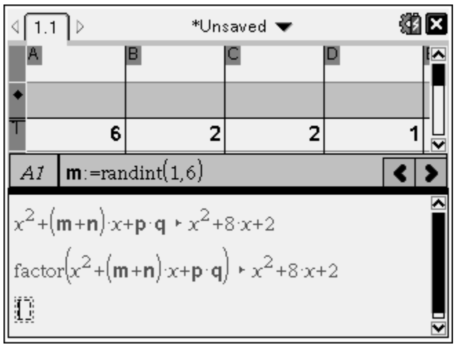


In the second maths box type: factor($x^2+(m+n)\cdot x+p\cdot q$)

The most common form of the factor command is:

factor(expression,variable)

Example: factor(x^2 + 6x + 2) will not show a factorised output as the omission of the variable reduces the output to integer factors, however factor(x^2 + 6x + 2,x) would be factorised using irrational factors.



The quadratic expression is random and based on the dice rolls simulated in the spreadsheet. The way the factor command has been set up in this situation, only rational factors will be displayed.

Switch control back to the spreadsheet by pressing **CTRL + Tab** or by using the mouse.

Note:

If cells A1 to D1 have disappeared, they may be hidden due to

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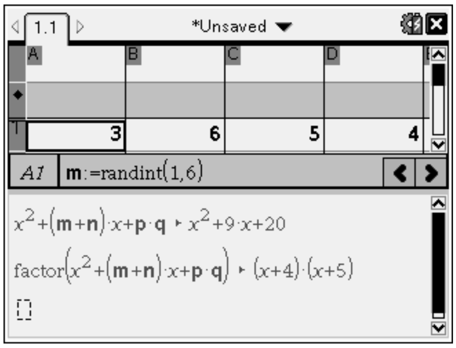
the reduced number of rows displayed on the screen.

With the focus on the spreadsheet, navigate to cell A1 and the simulated dice rolls will be displayed.

In the spreadsheet press **CTRL + R** to simulate another dice roll with subsequent expanded and potentially factorised quadratic trinomials.



Probability through simulation

1. To get a sense of the probability of rolling a quadratic that will factorise over the rational number field, simulate 100 rolls of the dice. Watch the notes page and count how many of the quadratic equations factorise. Write your answer as a probability or percentage.

Results will vary but typically vary between 7% and 27% with the actual probability close to 17.4%.

Your response to question one provides an indication of the actual answer. More simulations will produce a better approximation to the actual answer. The sample space for the dice sum (x coefficient) and dice product (constant) will help calculate the theoretical answer to this problem, a lattice diagram is one way to illustrate the sample space.

2. Complete the lattice diagram below for the sum of the two dice; some of the answers have been completed. The lattice diagram represents all the possible coefficients for x, in $y = x^2 + bx + c$

Dice Sum	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

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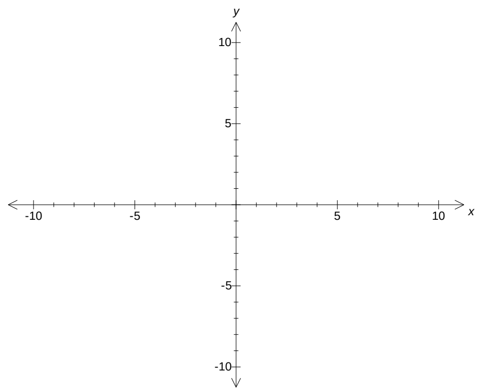
3. Complete the lattice diagram below for the product of the two dice; some of the answers have already been completed. The lattice diagram represents all the possible constants in $y = x^2 + bx + c$

Dice Prod.	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

The two lattice diagrams show the individual possibilities for the coefficient (b) and the constant (c). For each coefficient there are 36 permutations making a total of 36 x 36 = 1296 possible equations to be factorised. Some of the equations are repeated, systematic exploration combined with some thinking and reflection will drastically reduce the time taken to explore which of these 1296 combinations factorise.

4. Suppose a [1] and a [1] are rolled on the first pair of dice to give an x coefficient of 2. The equation would be: $y = x^2 + 2x + c$ Two more dice must be rolled, the product of which form the constant (c). The lattice diagram for the product can be used to write down all the possible quadratics: $y = x^2 + 2x + 1$, $y = x^2 + 2x + 2$ ...
$y = x^2 + 2x + 36$

a. Write down all the quadratics that can be formed.



$y = x^2 + 2x + 1$ $\quad$ $y = x^2 + 2x + 2$
$y = x^2 + 2x + 3$ $\quad$ $y = x^2 + 2x + 4$
$y = x^2 + 2x + 5$ $\quad$ $y = x^2 + 2x + 6$
$y = x^2 + 2x + 8$ $\quad$ $y = x^2 + 2x + 9$
$y = x^2 + 2x + 10$ $\quad$ $y = x^2 + 2x + 12$
$y = x^2 + 2x + 15$ $\quad$ $y = x^2 + 2x + 16$
$y = x^2 + 2x + 18$ $\quad$ $y = x^2 + 2x + 20$
$y = x^2 + 2x + 24$ $\quad$ $y = x^2 + 2x + 25$
$y = x^2 + 2x + 30$ $\quad$ $y = x^2 + 2x + 36$

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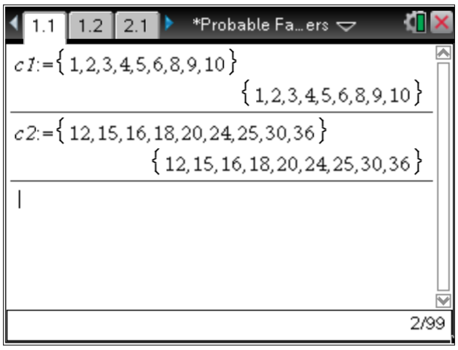
b. Determine which quadratics can be factorised.

$y = x^2 + 2x + 1$ $(x + 1)^2$

c. Graph each of the equations indicating which ones factorised successfully over the rational number field.

A fast way to produce the family of curves is to set c = {1, 2, 3, 4, 5, 6, 8, 9, 10, 12...36}.



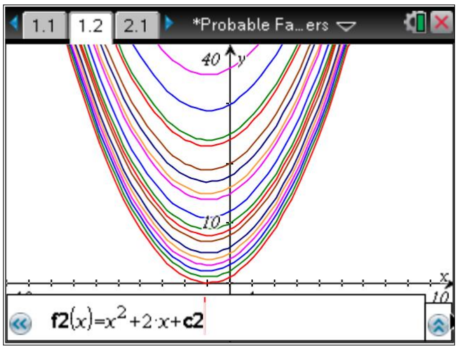
There is a limit of 16 values for a parameter such as c, when graphing a family of curves such as: $= x^2 + 2x + c$. There are 18 elements in *our* value for c. A quick way to get around this problem is to have C1 and C2.

C1:={1, 2, 3, 4, 5, 6, 8, 9, 10}

C2: = {12, 15, 16, 18, 20, 24, 25, 30, 36}

The only equation that factorises is:

$y = x^2 + 2x + 1$



This is a perfect square and is the only graph that touches the x axis.

d. Use the lattice diagram for the dice products to determine how many of the quadratics of the form: $y = x^2 + 2x + c$ will factorise.

The only equation that factorises over the rational number field is when c = 1. The only way this product can be produced is [1, 1], therefore should such an equation be produced there is only one chance it will factorise of the rational number field.

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5. Suppose a [5] and a [3] are rolled on the first pair of dice to give an x coefficient of 8. The equation would be: $y = x^2 + 8x + c$ Two more dice must be rolled, the product of which form the constant (c). The lattice diagram for the product can be used to write down all the possible quadratics: $y = x^2 + 8x + 1$, $y = x^2 + 8x + 2$ ... $y = x^2 + 8x + 36$

   a. Write down all the quadratics that can be formed.

$y = x^2 + 8x + 1$	$y = x^2 + 8x + 2$
$y = x^2 + 8x + 3$	$y = x^2 + 8x + 4$
$y = x^2 + 8x + 5$	$y = x^2 + 8x + 6$
$y = x^2 + 8x + 8$	$y = x^2 + 8x + 9$
$y = x^2 + 8x + 10$	$y = x^2 + 8x + 12$
$y = x^2 + 8x + 15$	$y = x^2 + 8x + 16$
$y = x^2 + 8x + 18$	$y = x^2 + 8x + 20$
$y = x^2 + 8x + 24$	$y = x^2 + 8x + 25$
$y = x^2 + 8x + 30$	$y = x^2 + 8x + 36$

   b. Determine which quadratics can be factorised.

$y = x^2 + 8x + 12$	$y = (x + 6)(x + 2)$
$y = x^2 + 8x + 15$	$y = (x + 5)(x + 3)$
$y = x^2 + 8x + 16$	$y = (x + 4)^2$

   c. Graph each of the equations indicating which ones factorised successfully over the rational number field

The only three equations that factorise:

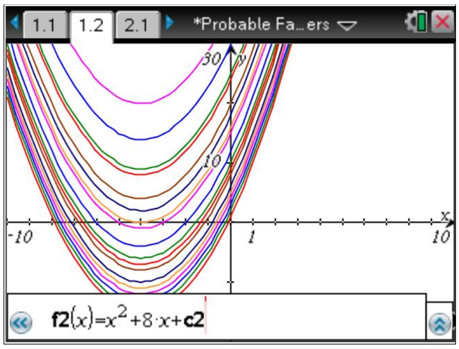
$y = x^2 + 8x + 12$

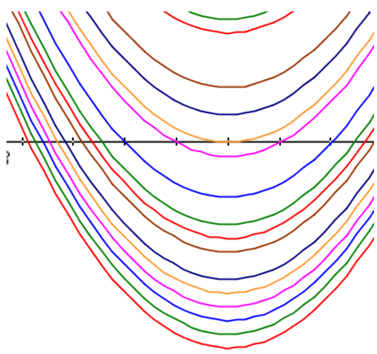
And

$y = x^2 + 8x + 15$

And

$y = x^2 + 8x + 16$





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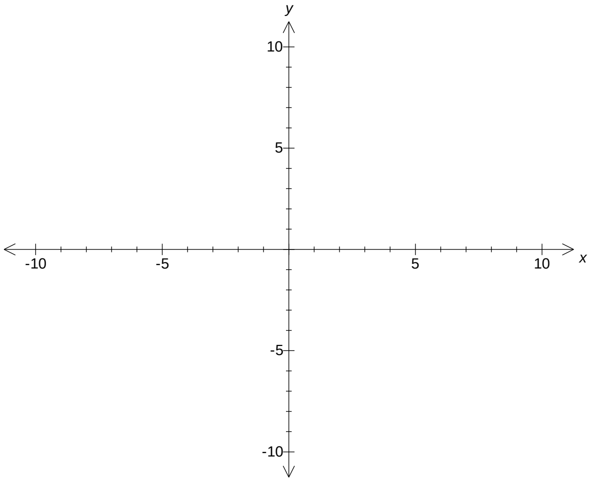
d. Consider both sets of graphs and observe the value for the constant where the family of graphs no longer cross the x axis.

i. Comment on your findings.



For equations: $y = x^2 + 8x + c$

- There are 6 equations that do not cross the x axis and therefore do not factorise over the real number field.
- There are 12 equations that cross or touch the x axis but only 3 of them have rational x intercepts and therefore rational factors.
- As per the previous set of equations, it is the 'perfect square' that is the 'last' of the sequence of graphs to cross the x axis.
- It is the 'perfect square' concept that allows students to progress rapidly through the remainder of the activity. It reduces enormously the number of equations that need to be investigated for factorising. It also yields a simple method for investigating factorising over the real number field.

ii. Use this to predict or hypothesise the value of the constant (c) such that the family of graphs $y = x^2 + 10x + c$ will no longer cross the x axis.

For equations: $y = x^2 + 10x + c$ the value for c will be 25.
This corresponds to: $y = (x + 5)^2$

iii. Check your hypothesis with your teacher and discuss how this can reduce the number of possible equations to be explored in this investigation.

It is important for students to notice the 'perfect square' at this point. If students have not yet identified the relationship, draw the family of graphs for:

$$y = x^2 + 10x + c$$

Followed by:

$$y = x^2 + 12x + c$$

Students should write a description for identifying the value of c given the x coefficient. (half and square) This reduces the number of equations that need to be explored from the dice product lattice diagram.

iv. What other dice combinations are possible to produce an x coefficient of 8?

The dice combinations provided in the question was [5, 3]. Other dice combinations to produce an x coefficient of 8 include: [6, 2] and [4, 4]; however students may

also include [2, 6] and [3, 5]
The purpose of the question is to have students realise that upon determining the number of successful outcomes for $y = x^2 + 8x + c$ they can complete more than just the [5, 3] combination in the lattice diagram.

v. Use the "dice product" lattice diagram to determine how many of the quadratic equations with an x coefficient of 8 will factorise?

The constant in the equation: $y = x^2 + 8x + 12$ can be produced with the following dice products: [6, 2], [2, 6], [3, 4] and [4, 3]. Therefore there are 4 possibilities that a quadratic equation of the form: $y = x^2 + 8x + 12$ will factorise over the rational number field.
The constant in the equation: $y = x^2 + 8x + 15$ can be produced with the following dice products: [5, 3] and [3, 5]. Therefore there are 2 possibilities that a quadratic equation of the form: $y = x^2 + 8x + 15$ will factorise over the rational number field.
The constant in the equation: $y = x^2 + 8x + 16$ can be produced with the following dice product: [4, 4]. Only 1 possibility exists for a quadratic equation of the form: $y = x^2 + 8x + 16$.
This means quadratics of the form: $y = x^2 + 8x + c$ have a total of 7 possible ways to be factorised using this random generation method.

6. The modified lattice diagram below includes a space to record the number of quadratics that factorise for the corresponding x coefficient. Complete the lattice diagram with each of the associated results.

Complete Analysis:

If the quadratic equation is: $x^2 + 2x + c$ rational factors will occur when c = 1. Checking the dice product lattice diagram this will only occur if (1, 1) is rolled. So the result for this quadratic equation is 1 occurrence.

If the quadratic equation is: $x^2 + 3x + c$ rational factors will only occur when c = 2. Checking the dice product lattice diagram this will only occur if a (1, 2) or (2, 1) combination is rolled. So the result for this quadratic equation is 2 occurrences.

If the quadratic equation is: $x^2 + 4x + c$ rational factors will only occur when c = 3 or 4. Checking the dice product lattice diagram a product of 3 or 4 can be produced by any of the following dice combinations: (1, 3), (3, 1), (2, 2), (1, 4) or (4, 1). So the result for this quadratic equation is 5 occurrences.

If the quadratic equation is: $x^2 + 5x + c$ rational factors will only occur when c = 4 or 6. Checking the dice product lattice diagram a product of 4 or 6 can be produced by any of the following dice combinations: (1, 4), (4, 1), (2, 2), (1, 6), (6, 1), (2, 3) or (3, 2). So the result for this quadratic equation is 7 occurrences.

If the quadratic equation is: $x^2 + 6x + c$ rational factors will only occur when c = 5, 8 or 9. Checking the dice product lattice diagram a product of 5, 8 or 9 can be produced by any of the following dice combinations: (1, 5), (5, 1), (2, 4), (4, 2) or (3, 3). So the result for this quadratic equation is 5 occurrences.

If the quadratic equation is: $x^2 + 7x + c$ rational factors will only occur when c = 6, 10 or 12. Checking the dice product lattice diagram a product of 6, 10 or 12 can be produced by any of the following dice combinations: (1, 6), (6, 1), (2, 3), (3, 2), (2, 5), (5, 2), (2, 6), (6, 2), (3, 4) or (4, 3). So the result for this quadratic equation is 10 occurrences.

If the quadratic equation is: $x^2 + 8x + c$ rational factors will only occur when c = 7, 12, 15 or 16, however a product of 7 cannot be obtained when two regular dice rolled. Checking the dice product lattice diagram a product of 12, 15 or 16 can be produced by any of the following dice combinations: (2, 6), (6, 2), (3, 4), (4, 3), (3, 5), (5, 3) or (4, 4). So the result for this quadratic equation is 7 occurrences.

If the quadratic equation is: $x^2 + 9x + c$ rational factors will only occur when c = 8, 14, 18 or 20, however a product of 14 cannot be obtained when two regular dice rolled. Checking the dice product lattice diagram a product of 8, 18 or 20 can be produced by any of the following dice combinations: (2, 4), (4, 2), (3, 6), (6, 3), (4, 5) or (5, 4). So the result for this quadratic equation is 6 occurrences.

If the quadratic equation is: $x^2 + 10x + c$ rational factors will only occur when c = 9, 16, 21, 24 or 25, however a product of 21 cannot be obtained when two regular dice rolled. Checking the dice product lattice diagram a product of 9, 16 or 24 can be produced by any of the following dice combinations: (3, 3), (4, 4), (4, 6), (6, 4) or (5, 5) So the result for this quadratic equation is 5 occurrences.

If the quadratic equation is: $x^2 + 11x + c$ rational factors will only occur when c = 10, 18, 24, 28, or 30, however a product of 28 cannot be obtained when two regular dice rolled. Checking the dice product lattice diagram a product of 10, 18, 24 or 30 can be produced by any of the following dice combinations: (2, 5), (5, 2), (3, 6), (6, 3), (6, 4), (4, 6), (5, 6) or (6, 5). So the result for this quadratic equation is 8 occurrences.

If the quadratic equation is: $x^2 + 12x + c$ rational factors will only occur when c = 11, 20, 27, 32, 35 or 36, however products of 11, 27, 32 and 35 cannot be obtained when two regular dice rolled. Checking the dice product lattice diagram a product of 20 and 36 can be produced by any of the following dice combinations: (4, 5), (5, 4) or (6, 6). So the result for this quadratic equation is 3 occurrences.

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Dice Sum & Factor Freq.	1	2	3	4	5	6
1	2 / 1	3 / 2	4 / 5	5 / 7	6 / 5	7 / 10
2	3 / 2	4 / 5	5 / 7	6 / 5	7 / 10	8 / 7
3	4 / 5	5 / 7	6 / 5	7 / 10	8 / 7	9 / 6
4	5 / 7	6 / 5	7 / 10	8 / 7	9 / 6	10 / 5
5	6 / 5	7 / 10	8 / 7	9 / 6	10 / 5	11 / 8
6	7 / 10	8 / 7	9 / 6	10 / 5	11 / 8	12 / 3

7. From the completed lattice diagram determine the theoretical probability for the number of quadratics that can be factorised in this dice simulation and compare this result to the estimated probability using the simulations.

   There are a total of 226 equations that factorise successfully.

   (1 + 2 x 2 + 3 x 5 + 4 x 7 + 5 x 5 + 6 x 10 + 5 x 7 + 4 x 6 + 3 x 5 + 2 x 8 + 3 = 226)

   There are 226 favourable outcomes from a total of 1296 possible. Probability of factorising a quadratic equation produced this way would be 226/ 1296 or approximately 17.4%.

Extension

What is the probability that a quadratic produced in the same way will factorise over the real number field?

Students can commence this investigation by changing the syntax of the 'factor' command to include ",x" which will include irrational factors.

If students have understood the significance of the 'perfect square' they should identify equations of the form: $y = x^2 + 8x + c$ can be factorised provided c $\leq$ 16. In the dice product lattice diagram there are 26 possibilities. Equations of the form: $y = x^2 + 10x + c$ can be factorised provided c $\leq$ 25. In the dice product lattice diagram there are 33 such possibilities.

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Dice Sum & Factor Freq.	1	2	3	4	5	6
1	2 / 1	3 / 3	4 / 8	5 / 14	6 / 17	7 / 23
2	3 / 3	4 / 8	5 / 14	6 / 17	7 / 23	8 / 26
3	4 / 8	5 / 14	6 / 17	7 / 23	8 / 26	9 / 30
4	5 / 14	6 / 17	7 / 23	8 / 26	9 / 30	10 / 33
5	6 / 17	7 / 23	8 / 26	9 / 30	10 / 33	11 / 35
6	7 / 23	8 / 26	9 / 30	10 / 33	11 / 35	12 / 36

There are 765 favourable outcomes from a total of 1296 possible. Probability of factorising a quadratic equation produced this way would be 765/ 1296 = 85/144 or approximately 59%.

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