Solving Exponential Equations

MATH NSPIRED

TEACHER NOTES

Math Objectives

- Students will numerically approximate the solutions to exponential equations.
- Students will graphically determine exact solutions to exponential equations using the functions $f(x) = a^x$ and $f^{-1}(x) = \log_a(x)$ and the composition $f \circ f^{-1}(x) = x$.
- Students will find the exact solution to exponential equations using algebraic techniques that employ the relationship $f^{-1} \circ f(x) = x$.
- Students will use appropriate tools strategically (CCSS Mathematical Practice).

Vocabulary

- exponential functions and equations
- logarithmic functions and equations
- inverse functions
- composition of functions

About the Lesson

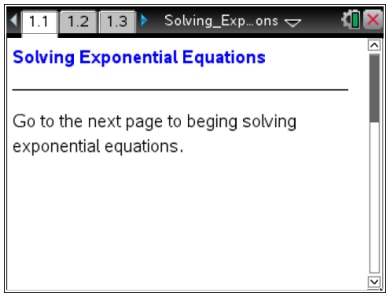
- This lesson involves numeric, graphical, and algebraic solutions to the equation $2^x = 3$. As a result, students will:
  - Analyze numeric patterns to predict an approximate solution.
  - Consider the graphs of both $f(x) = 2^x$ and $f^{-1}(x) = \log_2(x)$ to determine that $f(x) = 3$ precisely when $f^{-1}(3) = x$.
  - Use the compositional relationship of $2^{\log_2 x} = x$ to solve the equation. That is, the solution to the equation $2^x = 3$ is $x = \log_2 3$, since $2^{\log_2 3} = 3$.
  - Consider composition in the opposite order, employing the fact that $\log_2(2^x) = x$ to solve an equation algebraically.
  - Use these techniques to solve similar equations.

TI-Nspire™ Navigator™

- Use Class Capture to share numeric spreadsheet data.
- Use Quick Poll to discuss student responses.
- Use Live Presenter to analyze graphs.

Activity Materials

Compatible TI Technologies: TI-Nspire™ CX Handhelds, TI-Nspire™ Apps for iPad®, TI-Nspire™ Software



Tech Tips:

- This activity includes screen captures from the TI-Nspire CX handheld. It is also appropriate for use with the TI-Nspire family of products including TI-Nspire software and TI-Nspire App. Slight variations to these directions may be required if using other technologies besides the handheld.
- Watch for additional Tech Tips throughout the activity for the specific technology you are using.
- Access free tutorials at http://education.ti.com/calculators/pd/US/Online-Learning/Tutorials

Lesson Materials:

Student Activity

Solving_Exponential_Equations_Student.pdf

Solving_Exponential_Equations_Student.doc

TI-Nspire document

Solving_Exponential_Equations_.tns

Discussion Points and Possible Answers

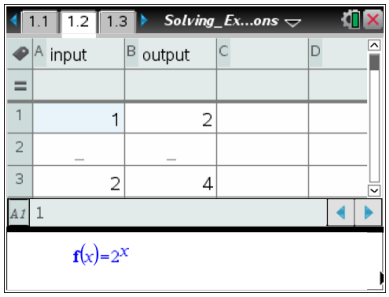
1. Estimate the solution to the equation $2^x = 3$ using the following numeric pattern:

   $2^1 = 2$
   $2^x = 3$
   $2^2 = 4$

   Sample answer: $x \approx 1.5$

TI-Nspire Navigator Opportunity: *Quick Polls (Open Response)*
See Note 1 at the end of this lesson.

Move to page 1.2.



2. The table shows inputs and outputs for the function $f(x) = 2^x$.

   a. Input your estimate from question 1 by entering it into cell A2. Input other values to get the output as close as possible to 3. Record your closest input and output below.

   Sample answer:

   $$f(1) = 2$$
   $$f(1.584) = 2.998$$
   $$f(2) = 4$$

   b. Is there an input value that results in an output value of exactly 3? Why or why not?

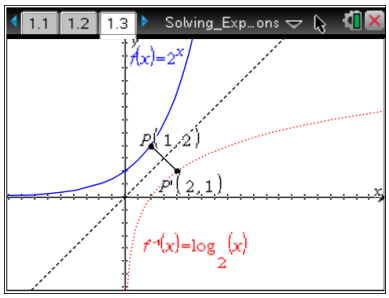
   Sample answer: Answers may vary. Some students may think there is no input value that results in 3 because 3 is not a multiple of 2. Students should try to enter more and more decimal places but they will never be able to get exactly 3 (unless they enter $\log_2 3$).

TI-Nspire Navigator Opportunity: *Class Capture*
See Note 2 at the end of this lesson.

Teacher Tip: If the approximate solution 1.584963 is input into the spreadsheet, the exact output appears to be 3. If necessary, briefly discuss as a class how the calculator is rounding the result.

Tech Tip: If students experience difficulty dragging a point, check to make sure that they have moved the cursor until it becomes a hand (☜) getting ready to grab the point. Then press ctrl ☜ to grab the point and close the hand (✊).

Move to page 1.3.



3. The graph of the function $f(x) = 2^x$ is shown along with its inverse $f^{-1}(x) = \log_2(x)$. Point P' is the reflection of point P over the line $y = x$.

   a. Suppose the coordinates of P are (1, 2). Write a logarithmic equation by substituting the coordinates of P' into the function $f^{-1}(x) = \log_2(x)$.

   Answer: $\log_2 2 = 1$

   b. Move point P' so that the input of the function $f^{-1}(x) = \log_2(x)$ is 3. According to the graph, what is the *approximate* solution to the equation $2^x = 3$? Why is this an approximate solution?

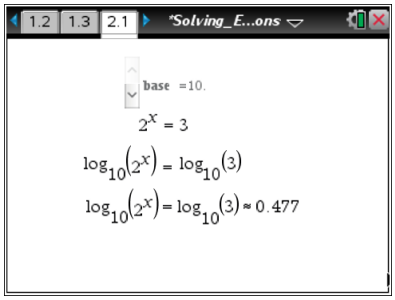
   Answer: When the input of f^{-1} is 3, the output is $\log_2 3 \approx 1.58496$. This means that the approximate solution is 1.58496.

   c. Recall that the composition of any function and its inverse always results in x. In other words, $f \circ f^{-1}(x) = f\left(f^{-1}(x)\right) = x$. Similarly, the composition of $f(x) = 2^x$ and $f^{-1}(x) = \log_2(x)$ results in the equation $2^{\log_2 x} = x$. Use this composition relationship to find the *exact* solution to the equation $2^x = 3$. What is the exact solution?

   Answer: $\log_2 3$

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 3 at the end of this lesson.

Move to page 2.1.



4. Solve the equation $2^x = 3$ by changing the base and reducing the left side of the equation to *x*. To change the base, click the up and down arrow. How do you find the exact solution using this algebraic method?

   Answer: Change the base to base 2, so $x = \log_2\left(2^x\right) = \log_2 3$, or more generally, $x = \log_b\left(b^x\right)$.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 4 at the end of this lesson.

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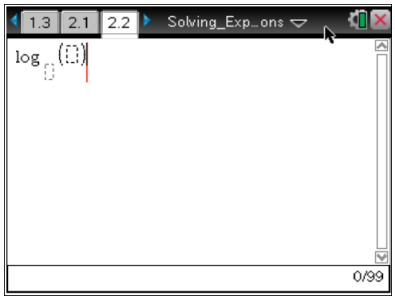


5. In questions 1 and 2 you found approximate solutions to the equation using numeric methods. You found exact solutions using graphical methods in question 3 and algebraic methods in question 4. Determine how close your estimates were by entering your exact answers in the Calculator page found on page 2.2. Do all three methods produce equivalent solutions?

   Answer: By calculating $\log_2 3$ directly in a calculator page, students should be able to get an approximate answer of 1.58496.

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 5 at the end of this lesson.

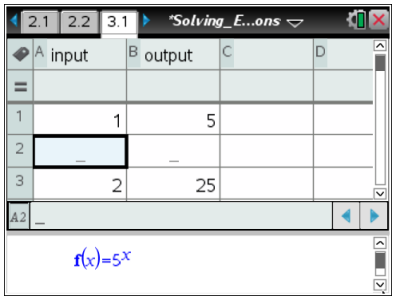
6. Estimate the solution to the equation $5^x = 7$ using the following numeric pattern:

   $5^1 = 5$
   $5^x = 7$
   $5^2 = 25$

   Sample answer: $x \approx 1.25$

TI-Nspire Navigator Opportunity: *Quick Polls*
See Note 6 at the end of this lesson.

Move to page 3.1.



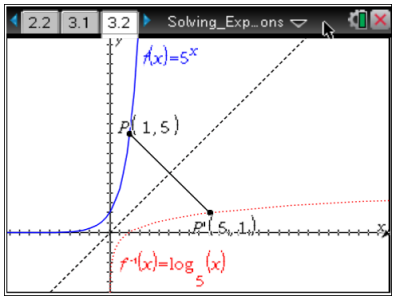
7. The table shows inputs and outputs for the function $f(x) = 5^x$.

   Input your estimate from question 6 by entering it into cell A2. Input other values to get the output as close as possible to 7. Record your closest input and output below.

   Sample answer:

$$f(1) = 5$$
$$f(1.209) = 6.999$$
$$f(2) = 25$$

Move to page 3.2.



8. The graph of the function $f(x) = 5^x$ is shown along with its inverse $f^{-1}(x) = \log_5(x)$. Point P' is the reflection of point P over the line $y = x$. Move point P' so that the input of $f^{-1}(x) = \log_5(x)$ is 7. According to the graph, what is the *approximate* solution to the equation $5^x = 7$?

   Answer: When the input of f^{-1} is 7, the output is $\log_5 7 \approx 1.20906$. This means that the approximate solution is 1.20906.

Move to page 4.1.

9. Solve the equation $5^x = 7$ by changing the base and reducing the left side of the equation to *x*. To change the base, click the up and down arrow. How do you find the exact solution using this algebraic method?

   Answer: You must change the base to base 5, so that you have $x = \log_5\left(5^x\right) = \log_5 7$.

TI-Nspire Navigator Opportunity: *Quick Polls (Open Response)*
See Note 7 at the end of this lesson.

Move to page 4.2.

10. Determine how close your estimates from questions 7 and 8 were by entering your exact answer from question 9 in this Calculator page. Do all methods produce equivalent solutions?

    Answer: By calculating $\log_5 7$ directly in a Calculator page, students should be able to get an approximate answer of 1.20906.

11. Use the algebraic method from questions 4 and 9 to find exact solutions to these equations.

    a. $3^x = 25$

    Answer: $\log_3 25$

    b. $8^x = 3$

    Answer: $\log_8 3$

    c. $4^x = 18$

    Answer: $\log_4 18$

Teacher Tip: Pages 5.1, 6.1, and 7.1 are set up with sliders if you choose to let students use these pages.

Teacher Tip: The last questions are provided as assessment. If you do not want students to use these last .tns pages (5.1, 6.1, 7.1), delete them before giving the file to students.

Tech Tip: To remove the last three pages, press ctrl ▲. Arrow to the pages to delete and press del on each page to clear. Resave with the same or another name.

TI-Nspire Navigator Opportunity: *Quick Polls (Open Response)*
See Note 8 at the end of this lesson.

Extension: Students have been solving equations of the form $a^x = b$, where $a,b \in \mathbb{R}, a > 1, b > 0$. Ask students why the conditions on *a* and *b* are necessary.

Wrap Up

Upon completion of the discussion, the teacher should ensure that students understand:

- Some exponential equations have exact decimal solutions and others do not.
- Non-decimal solutions can be approximated for these other equations.
- Non-decimal solutions can be approximated geometrically using the inverse relationship between exponential and logarithmic functions.
- Exact solutions can be found geometrically using the inverse relationship between exponential and logarithmic functions.
- Exact solutions can also be found algebraically using the inverse relationship between logarithmic and exponential functions and the compositional relationship between inverse functions.

Assessment

Question 11 could be used as an assessment item). Students could also be asked to solve problems that require multiple steps, such as: $3^x + 5 = 8$ or $2 \cdot 5^x = 7$

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Note 1

Question 1 *Quick Poll (Open Response):* Send an Open Response Quick Poll asking students to submit their answers to question 1.

Note 2

Question 2 *Class Capture*: Take a Screen Capture to allow students to share their inputs and outputs.

Note 3

Question 3 *Live Presenter*: Live Presenter can be used to show students how to grab and drag the point *P*' as well as to make sure students remember the relationship between the ordered pairs of two functions that are inverses.

Note 4

Question 4 ***Live Presenter:*** Live Presenter can be used to show students how to operate a minimized slider to change the base of the logarithm.

Note 5

Question 5 ***Live Presenter:*** Live Presenter can be used to show students how to enter the base and the argument for the logarithmic expression.

Note 6

Question 6 ***Quick Poll (Open Response):*** Send an Open Response Quick Poll asking students to submit their answers to question 6.

Note 7

Question 9 ***Quick Poll (Open Response):*** Send an Open Response Quick Poll asking students to submit their answers to question 9.

Note 8

Question 11 ***Quick Poll (Open Response):*** Send an Open Response Quick Poll asking students to submit their answers to questions 11a–11c.

TI-Nspire Navigator Tip: In order to type in logarithmic expressions correctly, students will need to press ctrl M to bring up the math template.