EXPLORATIONS™

Activity 14

Objectives

- Identify whether a slope field appropriately reflects a differential equation
- Determine whether a potential solution fits the slope field

Materials

- TI-84 Plus / TI-83 Plus
- Graph paper
- EULER program
- SLPFLD program

Using Slope Fields

Introduction

In this activity, you will practice interpreting slope fields by matching them up with differential equations. A slope field program for the graphing handheld can be used to generate slope fields for any differential equation of the form

$$\frac{dy}{dx} = h(x, y)$$

where $h(x, y)$ is any expression in terms of x and y. To use the program **SLPFLD**, enter **Y1** as the expression for $\frac{dy}{dx}$ in the Y= editor, and then go to the Home screen.

Select **EXEC SLPFLD** from the **PROGRAM Menu** to run the program.

Exploration

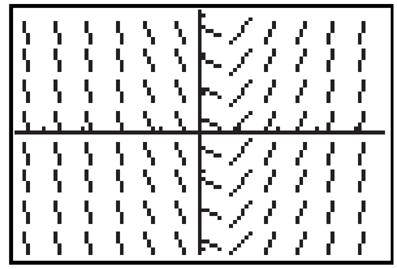
Use the following steps to create a slope field for the differential equation

$$\frac{dy}{dx} = 2x - 1$$

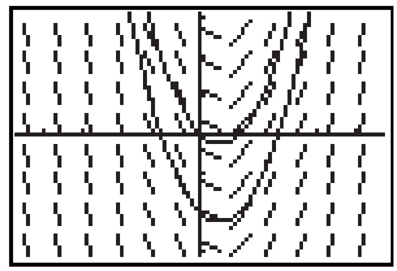
1. Enter **Y1=2X–1** in the **Y=** editor.
2. Go to the home screen.
3. Press [PRGM].
4. Scroll down and select **SLPFLD**.
5. Press [ENTER].

   You will see **prgmSLPFLD** on the calculation screen. Press [ENTER].

Running the program **SLPFLD** in the **ZDecimal** viewing window produces the slope field as shown in this screenshot.



You can graph one or more solution curves on top of this slope field as long as you do not change the window settings. Leave **Y1** unselected, enter both **Y2=X²−X** and **Y3=X²−X−2** in the **Y=** editor, and then press [GRAPH]. Your screen should match the one shown.



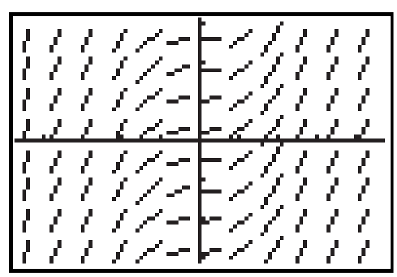
The simplest kind of differential equation is one of the form

$$\frac{dy}{dx} = f(x)$$

where $f(x)$ is an expression in terms of x only. In this case, you are looking for a function $y = F(x)$ having $f(x)$ as its derivative: $F'(x) = f(x)$. Such a function F would be called an *antiderivative* of the function f. Because the derivative is in terms of x only, all the line segments in a slope field that correspond to the same value of x should have the same slope. This means that you should see vertical columns of parallel line segments in the slope field because x is constant for a vertical line.

The differential equation

$$\frac{dy}{dx} = \ln(x^2 + 1)$$



produces the slope field output as shown in this screenshot from the **SLPFLD** program, using the **ZDecimal** viewing window.

Notice that within each vertical column, the line segments are parallel because the slope value depends on x only.

Many special but important differential equations are of the form

$$\frac{dy}{dx} = g(y)$$

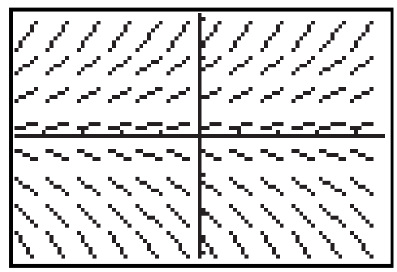
where $g(y)$ is an expression in terms of y only. For example, exponential functions are solutions to the classic differential equation

$$\frac{dy}{dx} = ky$$

where k is a constant. A positive value for k indicates *exponential growth;* a negative value for k indicates *exponential decay*. Because the derivative is in terms of y only, all the line segments in a slope field that correspond to the same value of y should have the same slope. This means that you should see horizontal rows of parallel line segments in the slope field because y is constant for a horizontal line.

The differential equation

$$\frac{dy}{dx} = \frac{y}{2}$$



produces the slope field output as shown in this screeshot from the **SLPFLD** program, using the **ZDecimal** viewing window.

Notice that within each horizontal row, the line segments are parallel because the slope value depends on y only.

For more general differential equations of the form

$$\frac{dy}{dx} = h(x, y)$$

check the axes and look for isoclines. An *isocline* is a curve along which all the segments of the slope field have the same slope (*iso* means "same," and *cline* means "slope").

For slope fields of more general differential equations, the slopes of the segments on the axes are usually easiest to check. For points along the x-axis, $y = 0$, and points along the y-axis, $x = 0$.

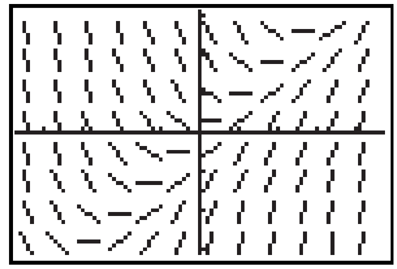
If the differential equation is

$$\frac{dy}{dx} = h(x, y)$$

then an isocline has the equation $h(x, y) = c$ where c is a constant.

The differential equation

$$\frac{dy}{dx} = x - y$$



produces the slope field output shown in this screenshot from the **SLPFLD** program, using the **ZDecimal** viewing window.

Look along the *x*-axis where $y = 0$ so that

$$\frac{dy}{dx} = x - 0 = x$$

You can see that the slopes have the same value as the *x*-coordinate. Look along the *y*-axis where $x = 0$ so that

$$\frac{dy}{dx} = 0 - y = {}^{-}y$$

You can see that the slopes have the opposite value of the *y*-coordinate.

Along any line associated with an equation in the form $x - y = c$ where *c* is constant, you should see only segments of slope *c*. For example, look along the line $y = x$ to verify that all the line segments are horizontal.

A slope field is particularly useful for checking the reasonableness of a symbolic solution to a differential equation that you have obtained from some other method (for example, by using a technique of antidifferentiation or by using separation of variables). By graphing a potential solution against the background of the slope field, you can see whether the solution curve fits.

Look at the slope field for

$$\frac{dy}{dx} = x - y$$

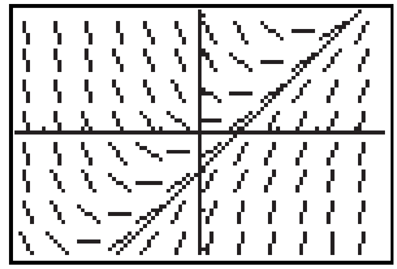
to see whether any solutions are in the form $y = mx + b$. Substituting into the differential equation, you find

$$m = x - (mx + b) = (1 - m)x - b$$

Subtracting m from both sides,

$$0 = (1 - m)x - (b + m)$$

The only way this could be true for all x would be if $0 = (1 - m)$ and $0 = (b + m)$. This would mean $m = 1$ and $b = {}^{-}1$. So the equation appears to have the solution $y = x - 1$. If you graph this against the slope field, you get the screenshot shown.



This looks reassuring. In fact, if you looked at the slope field first, then you might have guessed that $y = x - 1$ was a possible solution. The proof, of course, lies in verifying that $y = x - 1$ satisfies the differential equation.

For a line, $\frac{dy}{dx}$ is the slope, so $\frac{dy}{dx} = 1$.

Substituting this and $y = x - 1$ into the differential equation $\frac{dy}{dx} = x - y$, you obtain

$$1 = x - (x - 1) = x - x + 1 = 0 + 1 = 1$$

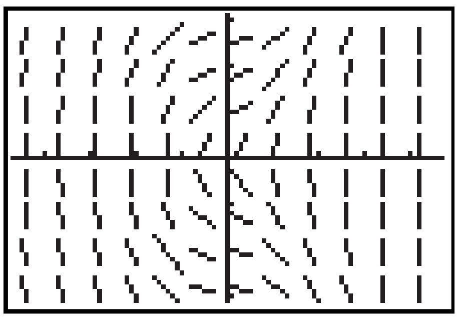
which checks out. Therefore, $y = x - 1$ is a solution.

Directions

Match each slope field in **Part I** with the corresponding differential equation in **Part II** *without* using the program **SLPFLD**. Then, use the program to check your solutions. Finally, match each slope field in **Part I** to the function in **Part III** that is a solution to the differential equation that corresponds to the slope field. Verify the solutions by substituting each into the differential equation.

Part I

1.



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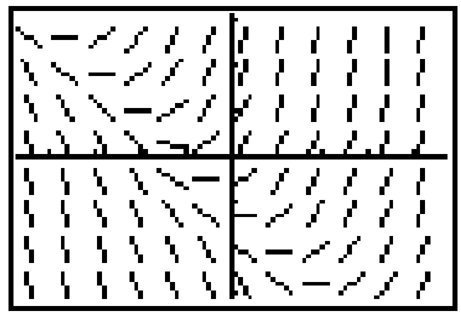

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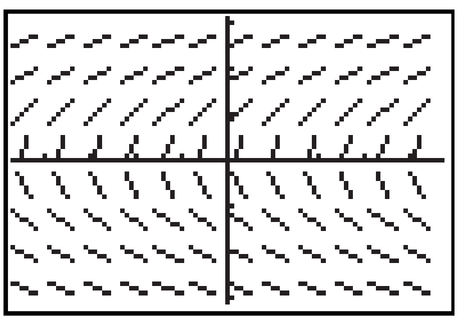

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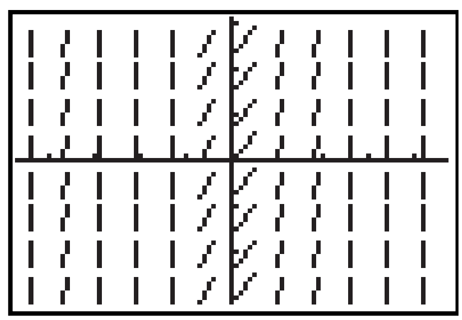

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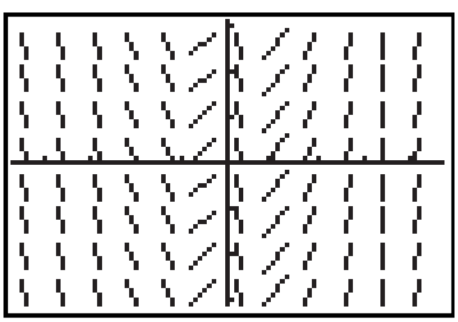

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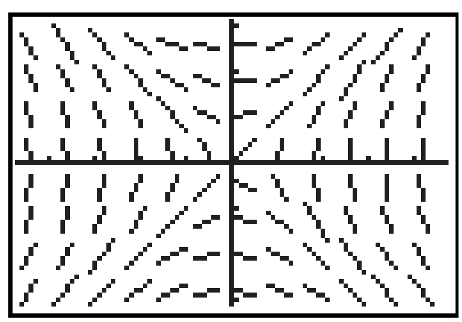

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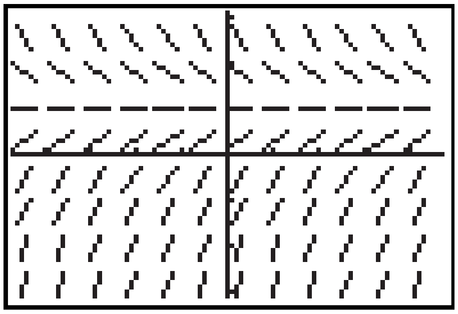

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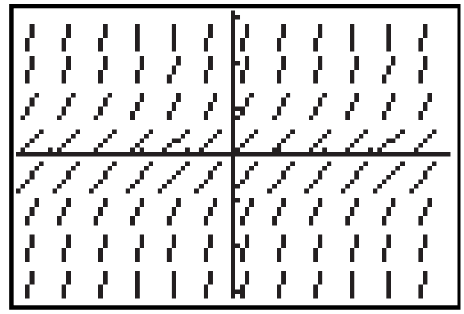

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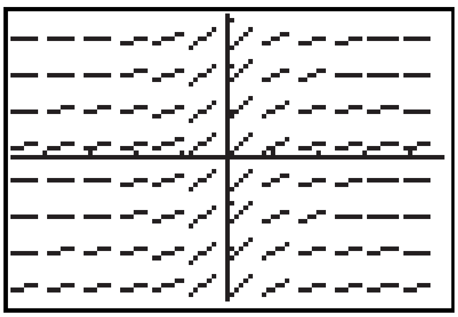

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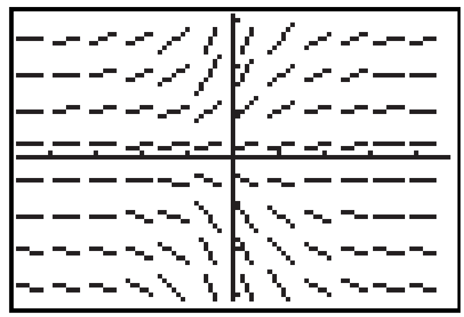

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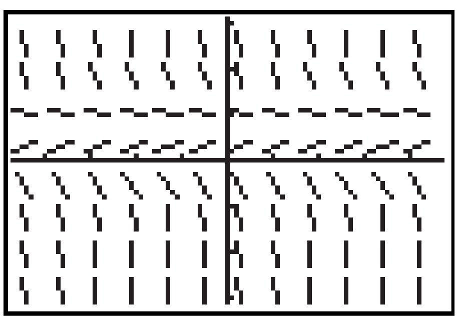

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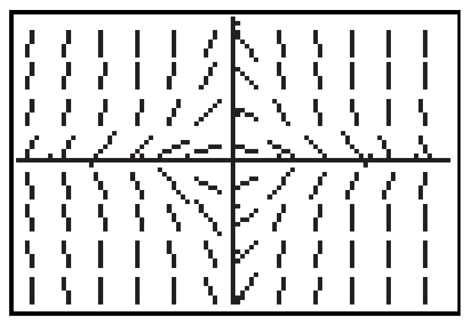

Part II

A $\frac{dy}{dx} = 2y(1-y)$

B $\frac{dy}{dx} = 1-y$

C $\frac{dy}{dx} = 3x^2+1$

D $\frac{dy}{dx} = \frac{x}{y}$

E $\frac{dy}{dx} = {}^{-}2xy$

F $\frac{dy}{dx} = x+y+1$

G $\frac{dy}{dx} = \frac{1}{y}$

H $\frac{dy}{dx} = \frac{1}{1+x^2}$

I $\frac{dy}{dx} = \frac{y}{1+x^2}$

J $\frac{dy}{dx} = 2x - \frac{1}{x}$

K $\frac{dy}{dx} = \frac{1+3x^2}{2y}$

L $\frac{dy}{dx} = 1+y^2$

Part III

m $y = \sqrt{x^2 + 1}$

n $y = \tan^{-1}(x)$

o $y = \tan(x)$

p $y = e^{\tan^{-1}(x)}$

q $y = x^2 - \ln(x) - 2$

r $y = {}^{-}\sqrt{2x + 3}$

s $y = e^{{}^{-}x^2}$

t $y = \dfrac{1}{1 + e^{{}^{-}2x}}$

u $y = \sqrt{x + x^3}$

v $y = 1 + e^{{}^{-}x}$

w $y = {}^{-}x - 2$

x $y = x^3 + x - 2$