Six Trigonometric Functions

Teachers Notes & Answers

7 8 9 10 **11** 12

TI-Nspire™ Investigation Student 45 min

Introduction

The three trigonometric functions taught most often in high school are: sine (sin), cosine (cos) and tangent (tan). There are however three other related trigonometric functions secant (sec), cosecant (cosec) and cotangent (cot). The origins of these terms can be attributed to Arabic mathematicians. The word 'sine' however comes from a mistranslation of the word 'chord'. In the unit circle sine is the measure of a semi-chord (half chord). Other terms such as cosine, cotangent and cosecant are all 90° out of phase with sine, tangent and secant as the prefix 'co' relates to 'complement'.

$$\cos(\theta) = \sin(90° - \theta) \qquad \cot(\theta) = \tan(90° - \theta) \qquad \csc(\theta) = \sec(90° - \theta)$$

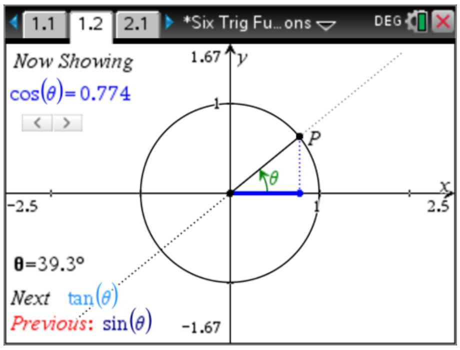
Imagine your calculator had just one trigonometric ratio, let's say: 'sine'. It is possible to use this one ratio to generate all the others! We can see from above that cosine is easy.

Example:

Suppose $\theta = 60°$, then according to the above relationships: $\cos(60°) = \sin(90° – 60°) = \sin(30°)$

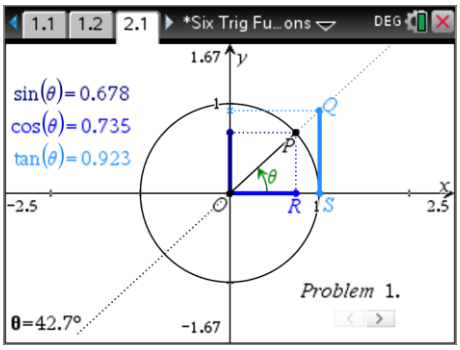
Similar Triangles

Open the TI-Nspire file "Six Trig Functions".



Page 1.2 contains a unit circle that can be used to explore all six trigonometric ratios. The example shown opposite is for cosine, this ratio is read from the x axis.

Use the slider beneath 'now showing' to explore each of the ratios on the unit circle. Drag point P around the circle to see how each one is calculated based on the size of the angle.



Navigate to Page 2.1 and select "Problem 1" using the slider.

Triangle ORP has sides OR (cosine), RP (sine) and OP (1 unit).

Triangle OSQ has sides OS (1 unit), SQ (tangent) and OQ (TBD)[1]

Angle POR is measured by θ and is common to both triangles.

Angle ORP and OSQ are both 90°

So by AAA, triangles ORP and OSQ are similar.

[1] TBD = To be determined (This length is determine later in this activity)

Author: P. Fox

Question: 1

Use the similar triangles ORP and OSQ to show that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$

$$\frac{QS}{PR} = \frac{OS}{OR}$$ By similar triangles

$$\frac{\tan(\theta)}{\sin(\theta)} = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Question: 2

The following exact values are known for sin(θ):

$$\sin(30) = \frac{1}{2} \qquad \sin(45) = \frac{\sqrt{2}}{2} \qquad \sin(60) = \frac{\sqrt{3}}{2}$$

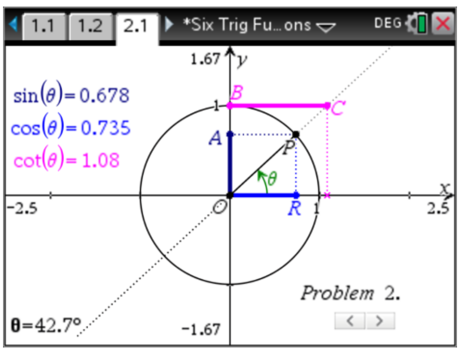
Use these exact values and the established relationships between sine, cosine and tangent to determine the corresponding exact values for cosine and tangent.

$$\begin{aligned} \cos(30) &= \sin(90-30) \\ &= \sin(60) \\ &= \frac{\sqrt{3}}{2} \end{aligned} \qquad \begin{aligned} \cos(45) &= \sin(90-45) \\ &= \sin(45) \\ &= \frac{\sqrt{2}}{2} \end{aligned} \qquad \begin{aligned} \cos(60) &= \sin(90-60) \\ &= \sin(30) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \tan(30) &= \frac{\sin(30)}{\cos(30)} \\ &= \frac{1}{2} \times \frac{2}{\sqrt{3}} \\ &= \frac{1}{\sqrt{3}} \end{aligned} \qquad \begin{aligned} \tan(45) &= \frac{\sin(45)}{\cos(45)} \\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} \\ &= 1 \end{aligned} \qquad \begin{aligned} \tan(60) &= \frac{\sin(60)}{\cos(60)} \\ &= \frac{\sqrt{3}}{2} \times \frac{2}{1} \\ &= \sqrt{3} \end{aligned}$$

On Page 2.1 select “Problem 3” using the slider.

Triangle OBC and OAP are similar. This similarity can be used to establish a relationship between cot(θ), sin(θ) and cos(θ) or collectively, tan(θ).



Author: P. Fox

Question: 3

Use the similarity of triangles OBC and OAP to write a rule relating cot(θ), sin(θ) and cos(θ), and hence express cot(θ) in terms of sin(θ) only.

$$\frac{BC}{AP} = \frac{1}{OA}$$ By similar triangles

$$\frac{\cot(\theta)}{\cos(\theta)} = \frac{1}{\sin(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

Expressing cot(θ) in terms of sin(θ):

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$
$$= \frac{\sin(90-\theta)}{\sin(\theta)}$$

Question: 4

Use the exact values for sin(θ) to calculate exact values for cot(30), cot(45) and cot(60).

$$\cot(30) = \frac{\sin(90-30)}{\sin(30)} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$$

$$\cot(45) = \frac{\sin(90-45)}{\sin(45)} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{1} = 1$$

$$\cot(60) = \frac{\sin(90-60)}{\sin(60)} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Question: 5

Use the similarity of triangles OBC and OAP to write a rule relating cot(θ) to tan(θ).

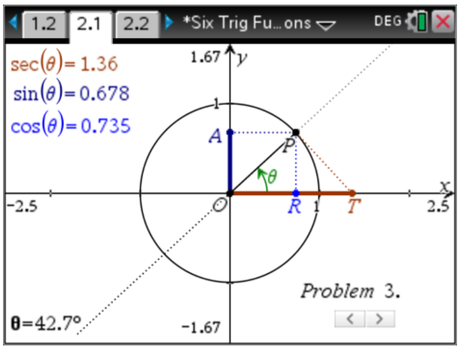
$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$ From Question 3 (similar triangles)

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$ From Question 1 (similar triangles)

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

On Page 2.1 select "Problem 3" using the slider.

Triangles OPR and OPT are similar as they both contain a right angle and have θ in common. This similarity can be used to establish a relationship between sec(θ), sin(θ) and cos(θ).



Author: P. Fox

Question: 6

Use the similarity of triangles OPR and OPT to write a rule relating sec(θ) to cos(θ) and hence write exact values for sec(30), sec(45) and sec(60).

$$\frac{OT}{OP} = \frac{1}{OR}$$ By similar triangles

$$\frac{\sec(\theta)}{1} = \frac{1}{\cos(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

Exact Values:

$$\sec(30) = \frac{1}{\cos(30)} = \frac{2}{\sqrt{3}} \qquad \sec(45) = \frac{1}{\cos(45)} = \sqrt{2} \qquad \sec(60) = \frac{1}{\cos(60)} = 2$$

Question: 7

Use the similarity of triangles OPR and OPT to show that length PT = tan(θ).

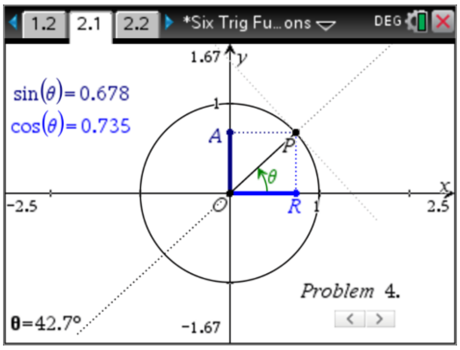
$$\frac{PT}{1} = \frac{PR}{OR}$$ By similar triangles

$$PT = \frac{\sin(\theta)}{\cos(\theta)}$$

$$PT = \tan(\theta)$$

Pythagoras Rules

Pythagoras's theorem can also be used to develop relationships between trigonometric functions. The following collection of problems relies on Pythagoras's theorem.



Question: 8

Use the slider to display information for problem 4. Use Pythagoras's theorem to write a relationship between sin(θ) and cos(θ). Check your answer using a range of angles.

$$OR^2 + RP^2 = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

Author: P. Fox

Question: 9

Use the slider to return to problem 3 and show that: $1+\tan^2(\theta)=\sec^2(\theta)$.

Hint: Use your result from Question 7.

$$OP^2+PT^2=1$$

$$1+\tan^2(\theta)=\sec^2(\theta)$$

Question: 10

Use the slider to return to problem 2, use similar triangles to show that length OC is equal to cosec(θ) and hence show that: $1+\cot^2(\theta)=\csc^2(\theta)$.

$$\frac{OC}{OP}=\frac{BC}{AP}$$ By similar triangles

$$OC=\frac{BC\times OP}{AP}$$

$$OC=\frac{\cot(\theta)\times 1}{\cos(\theta)}$$

$$OC=\frac{\cos(\theta)}{\sin(\theta)}\times\frac{1}{\cos(\theta)}$$

$$OC=\csc(\theta)$$

Proof of identity:

$$OB^2+BC^2=OC^2$$

$$1+\cot^2(\theta)=\csc^2(\theta)$$

Question: 11

Navigate to page 2.2 (calculator application) and enter the equation: $1+\cot^2(\theta)=\csc^2(\theta)$.

How does the calculator respond? Discuss.

The calculator display "True" but also a warning sign! The warning relates to the domain to alert users that some of the simplifications made may include a division by zero for specific cases.

Question: 12

Select Problem 3 on page 2.1. From Question 7 we know that PT = tan(θ). We can also see that PR = sin(θ) and that RT = sec(θ) – cosθ. From Pythagoras's theorem we can therefore conclude:

$$\left(\sec(\theta)-\cos(\theta)\right)^2+\sin^2(\theta)=\tan^2(\theta)$$

Using the result from question 7, make an appropriate substitution for sec(θ) and show (by hand) that the above equation is true.

$$\left(\sec(\theta)-\cos(\theta)\right)^2+\sin^2(\theta)=\tan^2(\theta)$$

$$\left(\frac{1}{\cos(\theta)}-\cos(\theta)\right)^2+\sin^2(\theta)=\tan^2(\theta)$$

$$\frac{1}{\cos^2(\theta)}-1-1+\cos^2(\theta)+\sin^2(\theta)=\tan^2(\theta)$$

$$\frac{1-\cos^2(\theta)}{\cos^2(\theta)}=\tan^2(\theta)$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)}=\tan^2(\theta)$$

Author: P. Fox

Teacher Notes:

This worksheet has been produced using ‘degrees’ rather than radians. This has been done to reduce cognitive overload at this relatively early stage of their understanding of trigonometric functions.

The dynamic representation of all six trigonometric functions on Page 1.2 can be used for many purposes.

- Show the range of values for each function, for example: $-1 \leq \sin(\theta) \leq 1$ or $|\sec(\theta)| \geq 1$ ...
- Cyclic nature of all six trigonometric functions.
- Focus on the prefix ‘co’ so students can see (from the graphic) that $\text{cosec}(\theta) = \sec(90 - \theta)$, and similarly for $\cot(\theta)$ and cosine(θ). Aside from pronunciation, stating the function in its full term, rather than its abbreviation, helps students remember this connectedness.
- Students can make up their own relationships through a combination of similar triangles, Pythagoras’s theorem and sections of triangles. (Such as the one in Question 12). Once students have established their relationship is true ... give it to another student to prove or ‘simplify’.
- Once students have a good knowledge of the relationships between the trigonometric relationships, a simple table will help them recall the main ‘exact’ values. See the sequence below to help students construct the full set:

Step 1:

	0	30	45	60	90
	0	1	2	3	4

Step 2:

	0	30	45	60	90
	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$

Step 2:

	0	30	45	60	90
	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$

Step 4:

θ	0	30	45	60	90
$\sin(\theta)$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1

Step 5:

Since $\text{cosine}(\theta) = \sin(90 - \theta)$ ... add the cosine values by simply reversing from 90°. Then add in the tangent ratios by dividing sine by cosine.

Author: P. Fox