Concepts

The general method used to construct a slope field can be used to determine a numerical approximation to the solution of a differential equation. Euler's method is based on the idea of local linearity, that is, a differentiable function is essentially linear on small intervals. This method can be used to produce a set of straight-line segments that approximates the graph of the solution to the differential equation, and to provide a numerical approximation to a point on the solution curve.

Suppose we know the value of a function f and its derivative at a single point. We can use this information to approximate a small portion of the graph of f using a straight-line segment; the tangent line to the graph of f at that point.

Consider a differential equation and an initial condition: $y' = F(x, y)$, $y(x_0) = y_0$. The objective is to find approximate points on the solution curve at equally spaced numbers $x_0, x_1 = x_0 + \Delta x, x_2 = x_1 + \Delta x, \ldots$ where Δx is the step size. The differential equation is used to find the slope of the tangent line at each point, for example, the slope at (x_0, y_0) is $y' = F(x_0, y_0)$.

The approximate value of the solution to the differential equation when $x = x_1$ is

$y_1 = y_0 + \Delta x \cdot F(x_0, y_0)$

The approximate value of the solution to the differential equation when $x = x_2$ is

$y_2 = y_1 + \Delta x \cdot F(x_1, y_1)$

And, in general

$y_n = y_{n-1} + \Delta x \cdot F(x_{n-1}, y_{n-1})$

Course and Exam Description Unit

Section 7.5: Approximating Solutions Using Euler's Method

Calculator Files

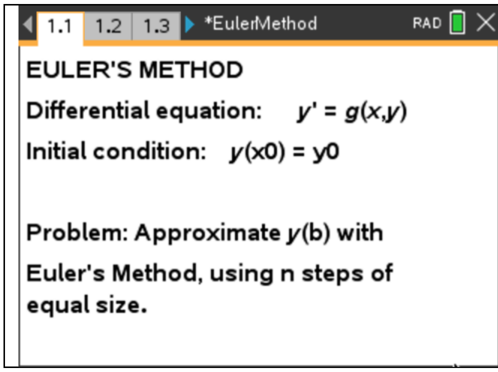
EulersMethod.tns

Using the Document

EulersMethod.tns: On page 1.2, the derivative $y' = g(x, y)$ is defined in a Math Box. The default definition for y' is $g(x, y) = \dfrac{xy}{4\sqrt{1 + x^2}}$. This expression can be changed by the user to allow for more in-depth and conceptual questions concerning Euler's Method. The initial condition, the endpoint x-value, and the number of Euler steps are also defined on page 1.2.

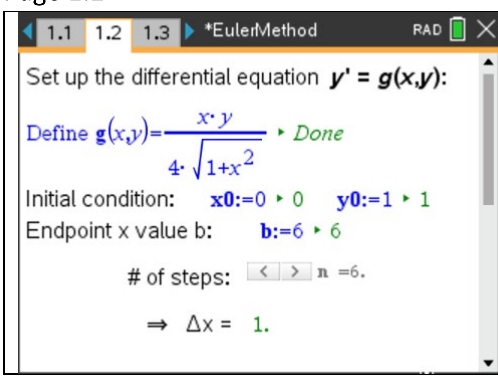
Page 1.3 is a Lists and Spreadsheet page that displays x_i, y_i, and $\Delta x \cdot g(x_i, y_i)$. Page 1.4 shows a graph of the points obtained using Euler's Method. The slider for n is used to change the number of steps and the slider for k is used to step through each Euler approximation

Page 1.1



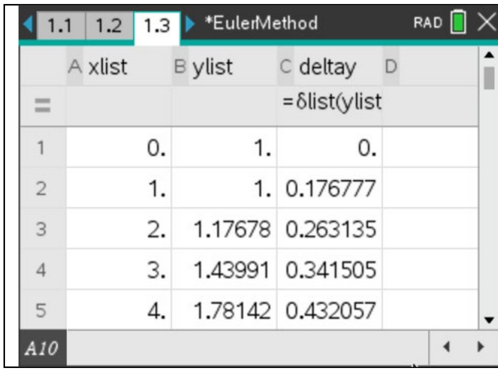
This page provides the notation use in this calculator file associated with Euler's Method. The initial value problem is $y' = g(x, y)$, $y(x_0) = y_0$. Euler's Method is used to approximate the value of the solution y at $x = b$ using n steps of equal size.

Page 1.2



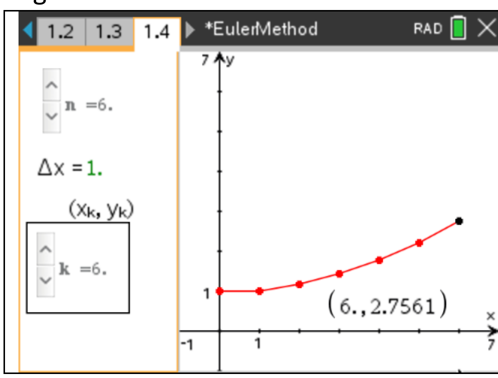
The derivative, given by the function $g(x, y)$, is defined in a Math Box. Note that this expression can be a function of both the variables x and y, and can be changed by the user to allow further exploration. The initial condition is also specified on this page, in two separate Math Boxes. And the value for the endpoint, b, is also specified here, in a Math Box. There is a slider used to set the number of Euler steps. The value of Δx is automatically computed and displayed.

Page 1.3



This Lists and Spreadsheet page displays x_i, y_i, and $\Delta x \cdot g(x_i, y_i)$ for the current differential equation and initial value, endpoint b, and number of steps n. Click in any cell to see a more accurate value in the entry line at the bottom of the screen.

Page 1.4



This page is a visualization of Euler's Method. Each step in the approximation procedure is plotted on the graph. The value of n (the number of steps) can be changed on this page. The variable k represents the step number. Use the slider to step through the solution. The coordinates of the kth step are displayed on the graph screen. Here, we can see that the approximation for $y(6)$ in this example is 2.7561.

Suggested Applications and Extensions

Use the default initial value problem, $y' = \frac{xy}{4\sqrt{1+x^2}}$, $y(0) = 1$, to answer questions 1-3. The values for x_0, y_0, b, and n can be set either in a Math Box or by using a slider. The default values are $x_0 = 0$, $y_0 = 1$, $b = 6$, and $n = 6$. The numerical approximations are given on page 1.3, a Lists and Spreadsheet page, and a visualization of the approximation is given on page 1.4.

1. Use Euler's Method to approximate $y(6)$ for each of the following values for n: (i) $n = 6$, (ii) $n = 12$, (iii) $n = 24$. Which value of n do you think produces the best estimate for $y(6)$? Why?
2. Use Euler's Method to approximate $y(-3)$ for each of the following values for n: (i) $n = 6$, (ii) $n = 12$, (iii) $n = 24$. Which value of n do you think produces the best estimate for $y(-3)$? Why?
3. Use Euler's Method to approximate $y(6)$ for $n = 6$. Use separation of variables to find an expression for y in terms of x. Add the graph of $y = f(x)$ on page 1.4 and compare it to approximation produced by Euler's Method. Use the graph of $y = f(x)$ to explain why the Euler approximation for $y(6)$ is an underestimate of the true value for $y(6)$.

Additional Problems

1. Use Euler's Method with $n = 4$ to estimate $y(2)$ where $y(x)$ is the solution to the initial-value problem $y' = 3y - x$, $y(1) = 0$.
2. Use Euler's Method with $n = 8$ to estimate $y(2)$ where $y(x)$ is the solution to the initial-value problem $y' = xy^2 - \frac{1}{4}x^2$, $y(0) = 1$. Consider each step in this Euler approximation. Explain why the estimate for $y(2)$ is so much larger than the estimate for $y(1.75)$.
3. Use Euler's Method with $n = 8$ to estimate $y(4)$ where $y(x)$ is the solution to the initial-value problem $y' = x + y$, $y(0) = 1$. Find y'' in terms of x and y, and use this expression to explain why this approximation is an underestimate or an overestimate for the true value of $y(4)$.
4. Use Euler's Method with $n = 8$ to estimate $y(2)$ where $y(x)$ is the solution to the initial-value problem $y' = \frac{y}{1+x^2}$, $y(0) = -1$. Use separation of variables to find an expression for y in terms of x. Graph $y = f(x)$ and the Euler approximation on the same coordinate axes. Explain why the first few Euler approximations are below the graph of $y = f(x)$ and the remaining approximations are above the graph of $y = f(x)$.
5. Use Euler's Method with $n = 8$ to estimate $y(\pi)$ where $y(x)$ is the solution to the initial-value problem $y' = \sin(x + y)$, $y(0) = 0$. Use $n = 16$ to estimate $y(\pi)$. Which estimate do you think is better? Why?
6. Use Euler's Method with $n = 8$ to estimate $y(-2)$ where $y(x)$ is the solution to the initial-value problem $y' = -x^2 y$, $y(0) = 1$. Use separation of variables to find an expression for y in terms of x. Graph $y = f(x)$ and the Euler approximation on the same coordinate axes. Find y'' and use this to explain why the Euler approximation for $y(-2)$ is an underestimate of the true value for $y(-2)$.
7. Let the function $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = y - x^2$ such that $f(0) = 1$.

(a) The function f has a critical point at $x = 1.67835$. What is the y-coordinate of this critical point?

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Use $\frac{d^2y}{dx^2}$ to determine whether the critical point found in part (a) is a relative minimum, relative maximum, or neither. Justify your answer.

(c) The function f has an inflection point at $x = \ln 2$. Use Euler's Method with $n = 10$ to estimate $y(1.67835)$ where $y(1) = 5 - e$. Is this approximation an overestimate or an underestimate. Justify your answer.