Kicking for Goals

Teacher Notes & Answers

7 8 9 10 11 **12**

TI-Nspire™ Investigation Student 75 min

Introduction

When it comes to kicking for goals, most players would rather be 'straight' in front of the goals and as close as possible. Opposition players try to defend the *region* around the goals. Where is this region? The region can be considered as a balance between distance and *shooting angle*. As a player runs towards the goals he or she may choose to pass the ball to a team member off to one side if they are closer to the goals. There is a compromise between proximity and shooting angle. This investigation focuses on optimising the shooting angle.

In extreme cases such as a corner penalty in soccer, or a mark near the behind post in Australian Rules Football (AFL), the shooting angle is 0°; commentators often refer to this as: "the player cannot see daylight between the posts". This investigation starts with the mathematically simpler soccer scenario where a player is not directly in front of the goals, no consideration is given to the variation in accuracy due to range. Part two of the investigation considers the more complicated scenario where a player moves along the curved boundary line to improve the shooting angle, a tactic used to an extreme by one AFL player!

Warm Up

Every athlete knows the importance of warming up prior to activity. The following '*exercises'* constitute an essential warm up before commencing this activity.

Question: 1.

a) Show that: $1+\tan^2(x)=\sec^2(x)$

Answer:

$$\frac{\cos^2(x)}{\cos^2(x)}+\frac{\sin^2(x)}{\cos^2(x)}=\sec^2(x)$$

$$\frac{1}{\cos^2(x)}=\sec^2(x)$$

b) Given $x=\tan(y)$ show that: $\dfrac{d\tan^{-1}(x)}{dx}=\dfrac{1}{1+x^2}$

Answer:

$$\frac{dx}{dy}=\sec^2(y)$$

$$\frac{dx}{dy}=1+\tan^2(y)$$

$$\frac{dy}{dx}=\frac{1}{1+x^2}$$

$$\frac{d(\tan^{-1}(x))}{dx}=\frac{1}{1+x^2}$$

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c) Given $\frac{d\ Tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$ use the chain rule to show that $\frac{d\ Tan^{-1}\left(\frac{a}{x}\right)}{dx} = \frac{-a}{a^2+x^2}$

$$\text{let } u = \frac{a}{x} \qquad y = \tan^{-1}(u)$$

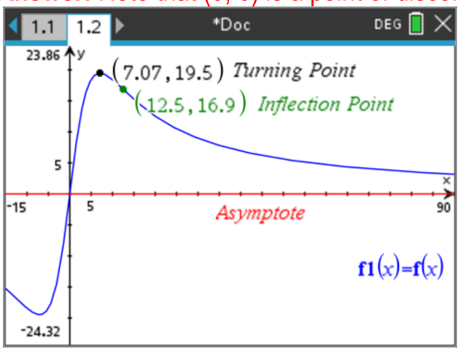
$$\frac{du}{dx} = \frac{-a}{x^2} \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

Answer: $\frac{dy}{dx} = \frac{-a}{x^2} \cdot \frac{1}{1+\frac{a^2}{x^2}}$

$$\frac{dy}{dx} = \frac{-a}{x^2+a^2}$$

d) Graph the function: $y = Tan^{-1}\frac{10}{x} - Tan^{-1}\frac{5}{x}$ with y measured in degrees. Identify key features of the graph including any intercept(s), turning point(s), points of inflection and asymptotes.

Answer: Note that (0, 0) is a point of discontinuity.



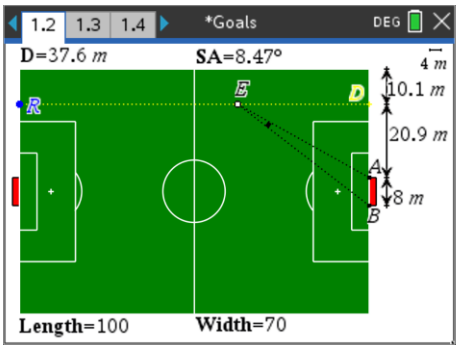
Shooting for Goal - Soccer

Sam is a mathematically inspired soccer coach, she is trying to teach the team the finer points of shooting for goal. The highest percentage shots are those with the maximum *shooting angle*. The *shooting angle* has the player at the vertex and the opening defined by the goal posts.

Open the TI-Nspire file: Goals

Navigate to page 1.2. This page contains an interactive layout of a soccer field. For the following questions, the length of the field is 100m, the width is 70m and the goals 8m wide.

Ellie is positioned at point E. She is planning on shooting for goals somewhere along the line RD which is 10m from the boundary. The goal posts are at A and B, so the shooting angle is ∠AEB.



Question: 2.

Let distance ED = x, determine an expression for the shooting angle ∠AEB in terms of x.

$$\measuredangle DEA = \text{Tan}^{-1}\left(\frac{21}{x}\right) \qquad \measuredangle DEB = \text{Tan}^{-1}\left(\frac{29}{x}\right)$$

Answer: $\measuredangle AEB = \measuredangle DEB - \measuredangle DEA$

$$\measuredangle AEB = \text{Tan}^{-1}\left(\frac{29}{x}\right) - \text{Tan}^{-1}\left(\frac{21}{x}\right)$$

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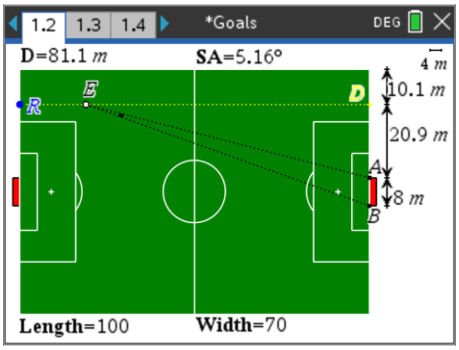
Move Ellie (point E) along the line RD. As Ellie moves parallel to the boundary watch the range of shooting angles produced. (SA)

The distance (D) and shooting angle (SA) are automatically captured. Navigate to page 1.3 to see the data.

Navigate to page 1.4 to see a graph of the data.

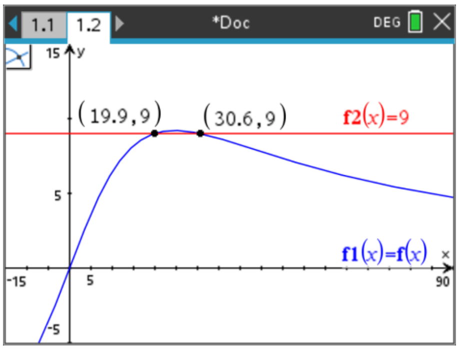
Check your answer to Question 2 by graphing your equation.

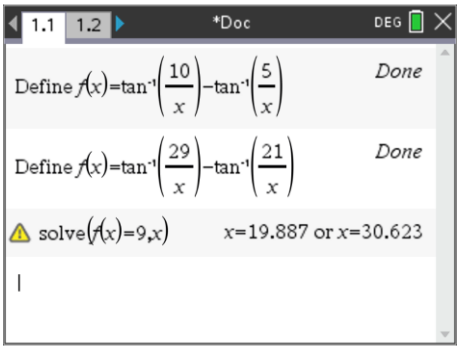


Question: 3.

Sam instructs Ellie to ‘take a shot’ as long as the shooting angle is greater than 9°, determine the range of positions along RD where Ellie can take her shot.

Answer: Solutions can be determined using the Graph or Calculator applications.

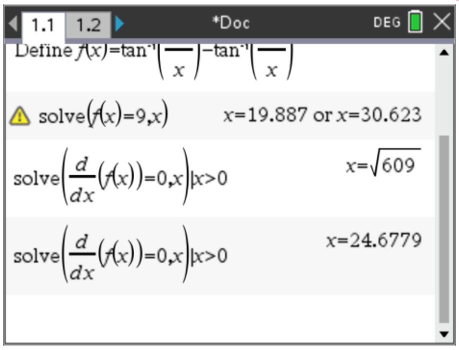




Question: 4.

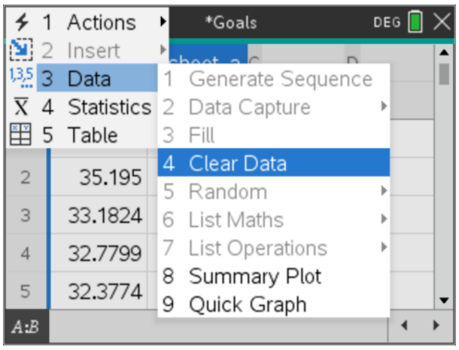
Use calculus to determine the maximum shooting angle and corresponding location.

Answer: Students should note that the angle lies between the two values obtained in Question 3.



The line RD is moved to a distance 15m from the boundary. Grab point R and move it so the line is approximately 15m from the boundary.

The original data must be cleared. Navigate to Page 1.3 and select the two columns, then use the menu to ‘**clear**’ the data.



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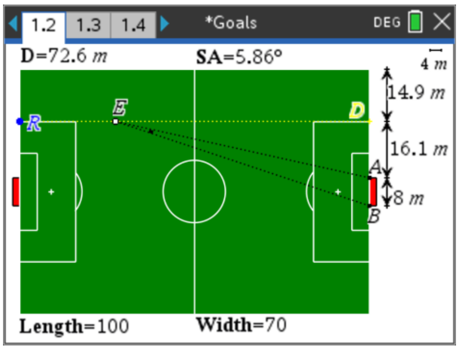
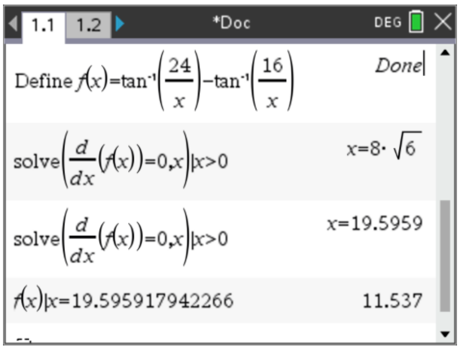
Drag E along RD (≈15m from boundary), determine a new equation and check your equation against the data. You will need to adjust the window settings on Page 1.4 to see the entire graph.

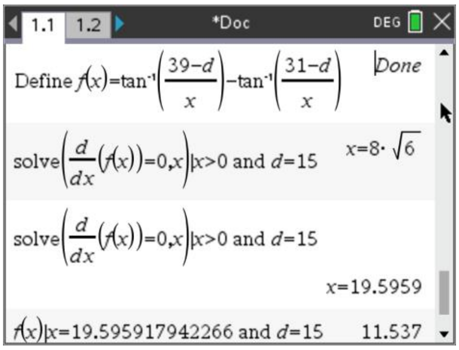


Question: 5.

Use calculus to determine the new maximum shooting angle and corresponding location.

Answer: Students can edit the previous function definition or even generalise by adding a parameter. Students may also comment on the fact that the shooting angle is wider, as expected since Ellie is now further from the boundary, so the goals should be slightly more 'opened up'.

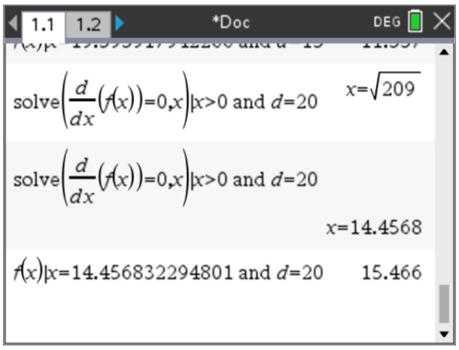


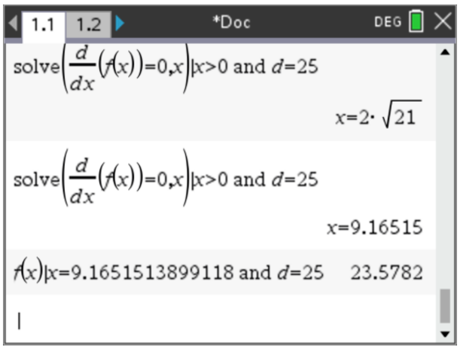


Question: 6.

Repeat this process with the line RD located 20m and then 25m from the boundary.

Answers: As Ellie gets closer and closer to the middle of the ground, it should be no surprise that the optimum shooting angle increases and the best shot is closer and closer to the goals. The two answers below verify thing thought process. At 20m in from the boundary the optimum shooting angle is 15.5° at a distance of 14.5m from the goal line. At 25m in from the boundary the optimum shooting angle increases to 23.6° at a distance of 9.17m.





Question: 7.

Determine the general location for Ellie's best shooting angle based on her distance from the boundary.
Example: If Ellie is 28m from the boundary, the rule should return the distance x (ED) for the optimum shooting angle, but not the angle itself.

Answer: If students have used a parameter in the previous two questions they should find this question easy.

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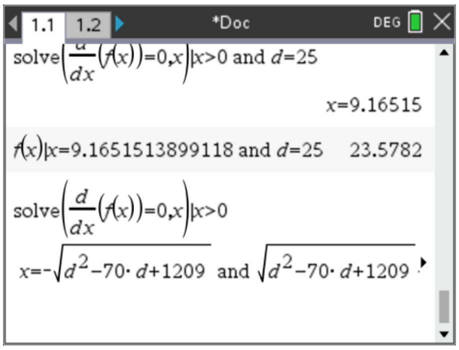


The expression: $\sqrt{d^2 - 70d + 1209}$ makes more sense when expressed as: $\sqrt{(d-39)(d-31)}$. This provides zeros (0m from the goal line) when d = 39 or d = 31 which corresponds to essentially running straight into the goals. ie: 31 < d < 39, x = 0.

AFL Kicking Angle

In an AFL match between Collingwood and Essendon in the early 1990s, Paul Salmon (Essendon) marked the ball near the point post. As per rule 20.5.1 (below), the umpire lined up the centre of the goals with the point post where the mark was taken. Paul walked back along this line with his opponent standing ‘on the mark’. Paul couldn’t walk back very far before bumping into the boundary fence, so he proceeded to walk around the boundary, distancing himself from the player on the mark in preparation for his natural run up. Paul just keep walking, almost all the way to the 50m arc. Paul managed to improve his shooting angle. One of Paul’s attributes was that he was an excellent long and straight kick of the ball. So the question is: “Did he shoot from the optimum location?”

20.5.1 Line of The Mark
Where a Player from the Attacking Team is Kicking for a Goal after being awarded a Mark or a Free Kick, the Kick shall be taken along a direct line from The Mark to the centre of the Attacking Team’s Goal Line, except in the following cases:

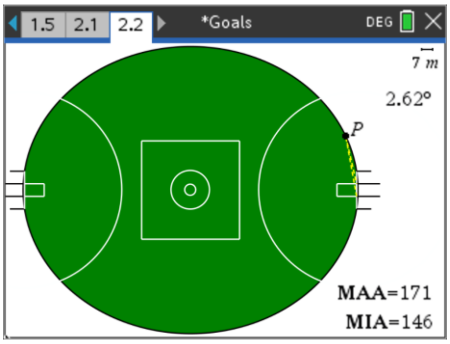
(a) Where the Mark or Free Kick is awarded within or on a line of the Goal Square, the Kick shall be taken from directly in front of the Goal Line from a spot horizontally across from where the Mark or Free Kick was awarded;

(b) Where the Kick will occur after the siren, the Player shall be entitled to approach The Mark from any direction, as long as the location of the Kick does not improve the angle to the goal posts.

MCG – Kicking Angle

Each AFL ground is slightly different. The MCG ‘oval’ is almost perfectly elliptical with a major axis 171m and a minor axis 146m. A scaled version of the oval is located on Page 2.2. Paul’s position is marked by point P. You can move point P along the boundary line and watch the angle change.

You can change the scale in the top right corner to 5 to get a much closer view of Essendon’s goals and Paul’s position.



Point P has been placed on an ellipse. If you move point P inside the goal line point P will end up ‘behind’ the goals which of course is not permitted in the game and geometrically result in much larger angles!

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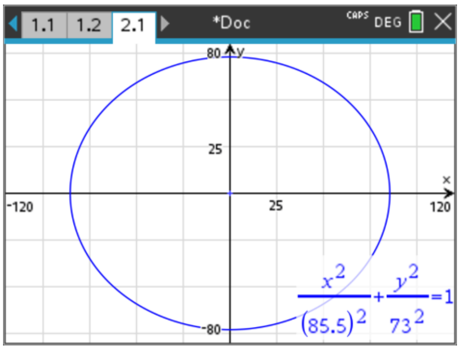
Question: 8.

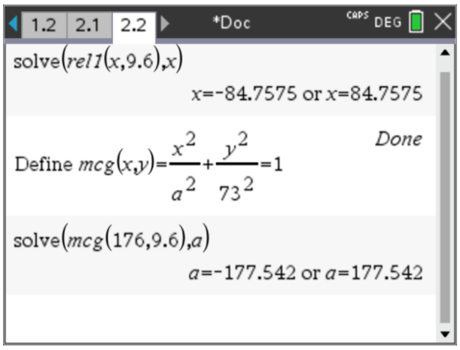
Let the centre of the ground represent the origin (Cartesian Plane). Determine the equation for the 'oval', given it is well approximated by an ellipse.

Answer: Note that the question only requires students to identify 'an' equation not the format for the equation, therefore students may also enter this equation in parametric form. A common error is not halving the dimensions provided since these are given as the overall length and width of the oval (ellipse) rather than from the centre.

Note: Students may complicate the question by making the 'oval' 171m between the goals, this is not the intention making it harder to determine the equation.



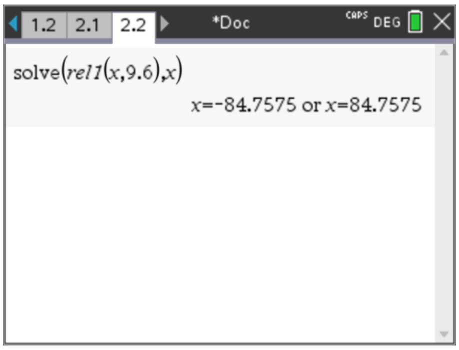
If students make the distance between the goal posts 171m and the width of the ground 73m, they can nominate the coordinates of the goal posts and then solve for the parameter 'a' to determine the equation for the ground. The end result is a slightly different answer, which will impact only slightly on subsequent answers.



Question: 9.

The goal posts are 6.4m apart, so too the point posts to the goal posts. Given the point posts (two outer posts) are on the boundary line (ellipse), determine the coordinates of each post.

Answer: Notice that the distance between the goal posts is slightly shorter than the overall ground length as the straight line connecting point post to point post cuts off the end of the ground. Overall field length: ≈ 169m



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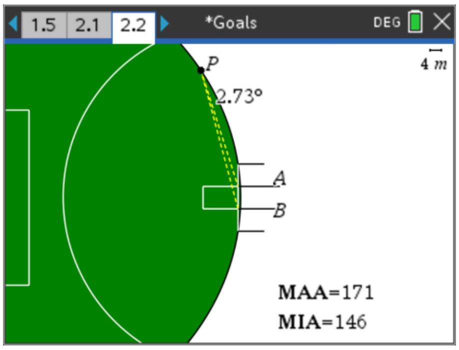
Question: 10.

Determine an expression for the goal kicking angle for any point along the playable boundary.
Note: The ground is symmetrical so the expression only needs to work along one side of the ground.

Answer: Required to find an expression for ∠APB



Coordinate of Point A (84.76, 3.2)

Coordinates of Point B (84.76, -3.2)

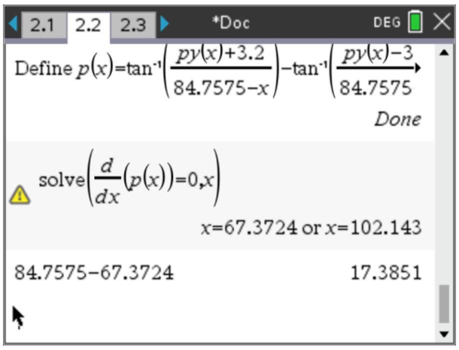
Coordinates of Point P $\left(x, \dfrac{73\sqrt{29241-4x^2}}{171}\right)$

Note: The ordinate is obtained by transposing the equation for the ellipse. For the expression for the angle below uses 'y' for simplicity.

$$\measuredangle APB = \tan^{-1}\left(\frac{y-3.2}{171-x}\right) - \tan^{-1}\left(\frac{y+3.2}{171-x}\right)$$

Question: 11.

Determine the location for the optimum shooting angle.



A maximum occurs when x = 67.37.

With the goal line at 84.76, that means Paul would be just 17.39m straight out from the goals, however he is on the boundary line.

Paul's coordinates: (67.37, 44.95)

Shooting Angle: 2.76°

Question: 12.

Given Paul kicked the ball from approximately 40m away from the goal line, determine if he managed to find the best location (within the rules) to take his kick. (Also assume that Paul is a very long and accurate kick of the ball).

Answer: Distance from Paul's location to the centre of the goals: $\sqrt{(84.76-67.37)^2+44.952} \approx 48.2m$

Paul was remarkably close to the ideal location. If Paul kicked at the 40m mark rather than at 48m the angle would have been ≈ 2.74°. A reduction of 8m in the kicking distance with a negligible change in angle means that if range accuracy considered, Paul's actual location was likely better than the calculus solution which does not account for accuracy over range.

Teacher Notes:

It is worthwhile taking students onto the school's football oval and have students walk around the boundary line until they sense they are at the optimum shooting angle. The first 20+m around the boundary line is relatively obvious as students start with 0° ('no daylight) and the angle *rapidly* opens up. By the time students are approximately 34m from goals, the angle has already opened up to 2.7°, very close to the maximum. The remaining walk around the boundary (48m from the centre of the goals) only provides and additional 0.4°.

Having students consider an 'accuracy' function based on distance will help students realise the complexity of the mathematics for what is a professional judgement by the player kicking the ball. Now players generally opt for a banana kick!

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