

# Great Expectations

## Answers

7 8 9 10 **11** 12



TI-30XPlus  
MathPrint™



Investigation



Student



50 min

## Introduction

Sometimes things just don't live up to their 'expectations'. In this activity you will explore three special dice and determine if they live up to their expectations. These dice may be regular in terms of their shape and form; however, each die does not contain the numbers one through to six. The numbers on each die are listed below:

- Red: {3, 3, 3, 3, 3, 6}
- Blue: {2, 2, 2, 5, 5, 5}
- Green: {1, 4, 4, 4, 4, 4}



A simple game is played between two people. Each player selects their own die: Red, Blue or Green. The dice are rolled, the player that rolls the highest number wins.

### Teacher Notes:

The dice used in this investigation are known as GRIME dice. Dr. James Grime created these dice which have seen a number of variations in order to generate other properties. There are other dice with non-transitive properties such as Efron Dice that are also worth investigating. This investigation starts with simulations to build a more cohesive relationship between probability and statistical concepts. So, whilst the end result could be obtained much quicker, the journey is arguably more important. The techniques being used also helps students build skills that can be applied to other problems.

### Question: 1.

Determine the Expected value for each die: Red, Blue and Green.

$$E(\text{Red}) = 3 \times \frac{5}{6} + 6 \times \frac{1}{6} = \frac{7}{2} \quad E(\text{Blue}) = 2 \times \frac{3}{6} + 5 \times \frac{3}{6} = \frac{7}{2} \quad E(\text{Green}) = 1 \times \frac{1}{6} + 4 \times \frac{5}{6} = \frac{7}{2}$$

### Question: 2.

Based on your calculations from Question 1, do you think the game is fair?

**Answer:** If the expected value is representative of the fairness ... then the game would be perfectly fair, however as students will soon discover, the expected value is not a suitable measure.

### Teacher Notes:

Consider two other dice: A: {1, 1, 1, 1, 1, 13} and B {2, 2, 2, 2, 2, 2}.  $E(A) = 3$ .  $E(B) = 2$ . In this situation B beats A in 5/6 rolls. It doesn't matter how much you win by, rather how frequently you win. This is why the expected value is not telling us the whole (or necessary) story.

## Creating the dice

The TI-30XPlus MathPrint calculator can generate random numbers.

To begin, press:



Select option 2: randint(

The syntax for this command is: **randint(lower, upper)**.

The lower limit is one (1). The upper limit is six (6).

To complete the command press:




To continue generating more random numbers keep pressing [enter].

Our 'special' dice do not have the numbers from 1 to 6. A combination of division and the 'integer' command will get us closer to our goal.

The integer command is located in the 'maths' menu. Press:

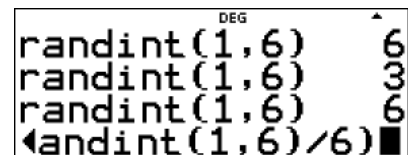
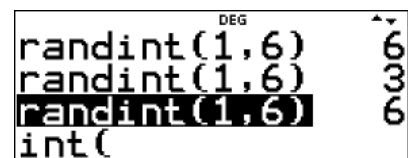
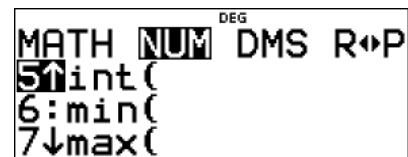
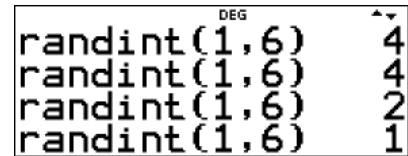


The randint(1,6) command needs to be inserted. It can be automatically inserted by copying and pasting!

Arrow up  to highlight the command.

Press  paste it.

We need to divide our random integers by 6. Press:



### Question: 3.

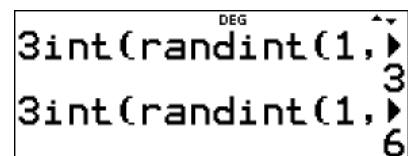
What sort of numbers are generated by the command combination:  $\text{int}(\text{randint}(1,6)/6)$ ?

You may need to press [ enter ] multiple times to get an idea.

**Answer:** Most students should see a combination of 0's and 1's. If students do not generate sufficient trials, they may not see a '1' occur. Students should also note that 0's occur more frequently than 1's.

Our simulator is almost complete. The full version to generate random rolls for the red die is as follows:

$$3 \times \text{int}(\text{randint}(1,6) \div 6) + 3$$



### Question: 4.

Explain how:  $3 \times \text{int}(\text{randint}(1,6) \div 6) + 3$  generates random numbers 3 and 6 in the same proportion as the red die.

**Answer:** randint(1,6) generates random numbers from 1 through to 6. These values are divided by 6, so randomly generated numbers from 1 to 5 produce values:  $0 \leq x < 1$ .  $\text{Int}(x)$  where  $0 \leq x < 1$  results in a 0. Therefore  $\text{int}(\text{randint}(1,6)) = 0$  for {1, 2, 3, 4, 5} and 1 when the random number is 6.

$\therefore \text{int}(\text{randint}(1,6) \div 6) = \{0, 0, 0, 0, 0, 1\}$ . So it follows that  $3 \times \text{int}(\text{randint}(1,6) \div 6) + 3 = \{3, 3, 3, 3, 3, 6\}$ , the red die.

**Question: 5.**

What set of random numbers are produced by the command combination:  
 $3 \times \text{int}(\text{randint}(1,6) \div 4) + 2$  ?

**Answer:** `randint(1,6)` generates random numbers from 1 through to 6. Division by 4 results in half of these values being less than 1. So `int(randint(1,6) ÷ 4)` generates: {0, 0, 0, 1, 1, 1}. Multiplying by 3 then adding 2 will result in: {2, 2, 2, 5, 5, 5} which is the same as the blue die.

**Question: 6.**

It's now time to play some games. Select an opponent and your respective colour. (Players must have different colours). Use the corresponding random generator combination for your dice on your calculator whilst your opponent uses theirs. Remember, you can copy and paste previous commands or simply press enter to recalculate. Record each die outcome, wins and losses in the table below.



**Answer:** Answers will vary as the results are 'random'.

[See teacher notes for additional information.]

<b>Game</b>	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
<b>Red</b>										
<b>Blue</b>										
<b>Winner</b>										
<b>Game</b>	#11	#12	#13	#14	#15	#16	#17	#18	#19	#20
<b>Red</b>										
<b>Blue</b>										
<b>Winner</b>										
<b>Game</b>	#21	#22	#23	#24	#25					
<b>Red</b>										
<b>Blue</b>										
<b>Winner</b>										

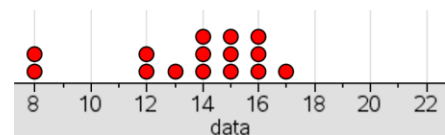
**Teacher Notes:**

Approximately 80% of students using the red die will win overall after 25 games. In many cases the overall win may not be significant enough to claim, with any confidence, that the win was anything more than chance. This type of discussion can be used to further enrich this activity. The discussion can lead into sampling distributions and hypothesis testing.

**Sampling Distribution:**

Each 'red die' student can be provided with a 'sticky dot' (red of course). Draw a horizontal axis on the board starting at 0 and ending at 25. Each red dice student places their dot corresponding to their number of wins therefore creating a dot plot.

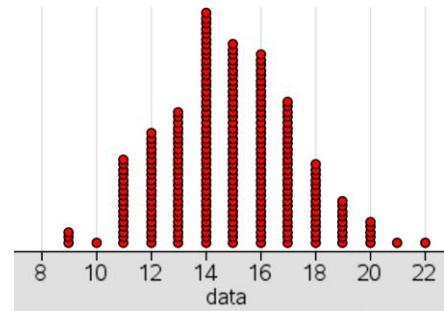
The dot-plot shown opposite is for a Sampling Distribution from 15 groups, each consisting of 25 games. The mean of this sampling distribution is 13.8. Red won 11 out of 15 samples.



The second dot-plot shows a Sampling Distribution for 250 groups with each group playing 25 games. Notice that the sampling distribution is approximately Normal (Central Limit Theorem).

In this sample of 250, RED won on 207 occasions. The mean of this sampling distribution is approximately 14.

How can we be more 'certain' about the mean of our sampling distribution, more samples or larger samples? The answer is larger samples! To see the impact of this, change to proportions rather than raw figures.



### Question: 7.

Based on your relatively small sample of 25 games, do you think the game is fair? Discuss.

**Answer:** Student answers will vary. Comments should reflect the size of the sample and the degree of confidence pertaining to their outcome.

### Teacher Notes:

The red dice only gives a slightly better probability of winning than the blue. In a sample of just 25 games the likelihood of students accepting the game as 'fair' is quite high, even though the game is actually unfair. This is referred to as a type II error, accepting the null hypothesis (the game is fair) when in fact the game is unfair.

More samples of 25 games will not reduce the likelihood of a type II error, playing more games in each sample will. This is why the calculator simulation increases the number of games being played to 50 in each sample. (Maximum list length).

Generally speaking the 25 games is not substantial enough to claim 'dominance'. Approximately 10% of students using the red dice may claim the red die is significantly better. It is highly unlikely (<0.004%) that students using the blue dice will claim that it is significantly better than the red.

## Calculator Simulation

It is possible to simulate 50 games at once!

The TI-30XPlus MathPrint has dynamic lists. To clear all the lists press:



The next step is to create a **sequence** for the red die. The sequence command can be found in the operations (OPS) menu.



The red dice rolls will be stored in List 1. Use the arrow keys to highlight list 1 (L1)

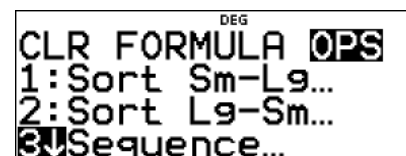
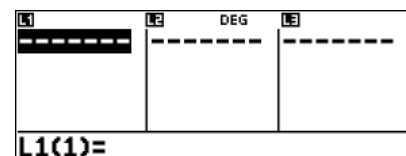


Enter the expression for the red die in the section for "EXPR IN"

$$3 \times \text{int}(\text{randint}(1,6) \div 6) + 3$$

The sequence can **Start** at 1 and **End** at 50 with a **Step Size** of 1. This will fill List 1 with 50 dice rolls.

**Note:** The calculator takes a few moments to generate the entire list.





As more values are entered into the calculator lists it will take longer to display the values, particularly when the lists are based on sequences and formulas. When all three lists are full, this can take a few seconds. Lists appear empty at first until they are re-populated.

An expression also needs to be entered for the blue die. Repeat the sequence process storing the blue dice rolls in List 2.

Notice that the red dice formula remains in the sequence command making it easy to edit and convert to the blue dice.

The screen opposite shows a sample of red dice rolls (List 1) and blue dice rolls (List 2).

L1	L2	DEG	L3
3	2		
3	2		
3	2		
6	2		
L2(1)=2			

It is possible to scroll through the list and see who wins each game, one game at a time. This process can be automated by using some 'clever' calculations:

- When you repeatedly calculate the root of a number between 0 and 1 the result will approach 1 (from below) but never reach it.
- When you repeatedly calculate the root of a number greater than 1, the result will approach 1 (from above) but will never reach it.
- When the red dice wins a larger number appears in List 1 compared to List 2. Calculating  $L1 \div L2$  would therefore produce a value greater than 1. When red loses the division would produce a result between 0 and 1.

Consider the situation where red rolls a 6 and blue rolls 2.  $L1 \div L2$  would result in a 3. Ideally we want this value to be a 1 (win for red). Since  $\sqrt{\sqrt{3}} \approx 1.3161$ , calculating the integer component only:  $int(\sqrt{\sqrt{3}}) = 1$ . The complete formula becomes:  $int(\sqrt{\sqrt{L1/L2}})$ .

To enter the formula in List 3 follow the instructions below.

Whilst in the Data screen, navigate to List 3:



Press:

stat-reg/distr

data

Navigate to "Add/Edit Frmla" (Add / Edit a formula)



Integer command:



Square-roots:



List names and division:



CLR	FORMULA	OPS
1: Add/Edit Frmla		
2: Clear L1 Frmla		
3: Clear L2 Frmla		

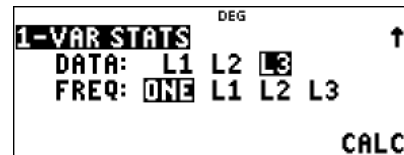
L1	L2	DEG	L3
3	5		0
3	2		1
3	2		1
3	2		1
L3=int(√(√(L1/L2)))			

Finally we can use the statistics menu to add up all the values (1's) in List 3.

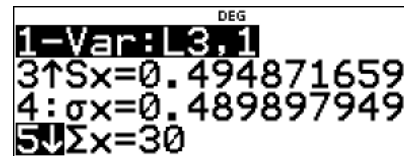
The statistics menu can find the sum of all the values in a list:



Select L3 (List 3) and Frequency as ONE, navigate down to calculate (CALC) and press:



Scroll down to item 5 to see the sum of all the values in the list. In this case the sum is equal to 30 signifying that the RED dice won 30 games out of 50.



Editing the sequence commands in List 1 and List 2 automatically generates a fresh set of random numbers. To reduce editing time, select the most recent sequence, check the formula and select SEQUENCE FILL. Remember to refresh **both** List 1 and List 2 sequences.

The formula in List 3 does not need to change, it will automatically re-calculate when values in List 1 or List 2 are changed.

### Question: 8.

Use the calculator to simulate a total of 100 games of Red vs Blue and record the overall results. From the results decide whether the red or blue dice provides a greater chance of winning.

**Answer:** Answers will vary with regards to the results of the simulation, however the sample is now substantial enough to provide a rough estimate for the probability of a win and to illustrate that the Red die is better than the Blue.

### Question: 9.

Determine an appropriate formula to generate the numbers on the green die: {1, 4, 4, 4, 4, 4}

**Check your formula with your teacher before proceeding.**

**Answer:**  $3 \times \text{int}(\sqrt{\text{randint}(1,6) \div 2}) + 1$ . Another alternative:  $3 \times \text{int}(\text{randint}(9,15) \div 10) + 1$

$\text{Randint}(1,6) \div 2$  produces: {0.5, 1, 1.5, 2, 2.5, 3} Calculating the square-root  $\approx$  {0.71, 1, 1.22, 1.41, 1.58, 1.73}

Calculating the integer quantity of all these: {0, 1, 1, 1, 1, 1}, multiplying by 3 and add 1: {1, 4, 4, 4, 4, 4}.

$\text{Randint}(9,15)$  produces: {9, 10, 11, 12, 13, 14, 15} so  $\text{int}(\text{randint}(9,15) \div 10)$  produces {0, 1, 1, 1, 1, 1}

Other answers are possible.

### Question: 10.

Simulate 100 games of Blue vs Green and record the overall results. From the results decide whether the blue or green dice provides a greater chance of winning.

**Answer:** Results will vary from simulation to simulation, however 100 trials usually enough to determine that the Blue die is better than the Green.

### Question: 11.

Based on your results to date; is Green more or less likely to beat Red? Discuss.

Thinking non-transitively, it would seem logical that since Red beats Blue and Blue beats Green ...  $R > B > G$  then it would follow that Red should beat Green. At this point of course students will probably not be aware of the non-transitive property, these dice are a little bit like Rock – Paper – Scissors.

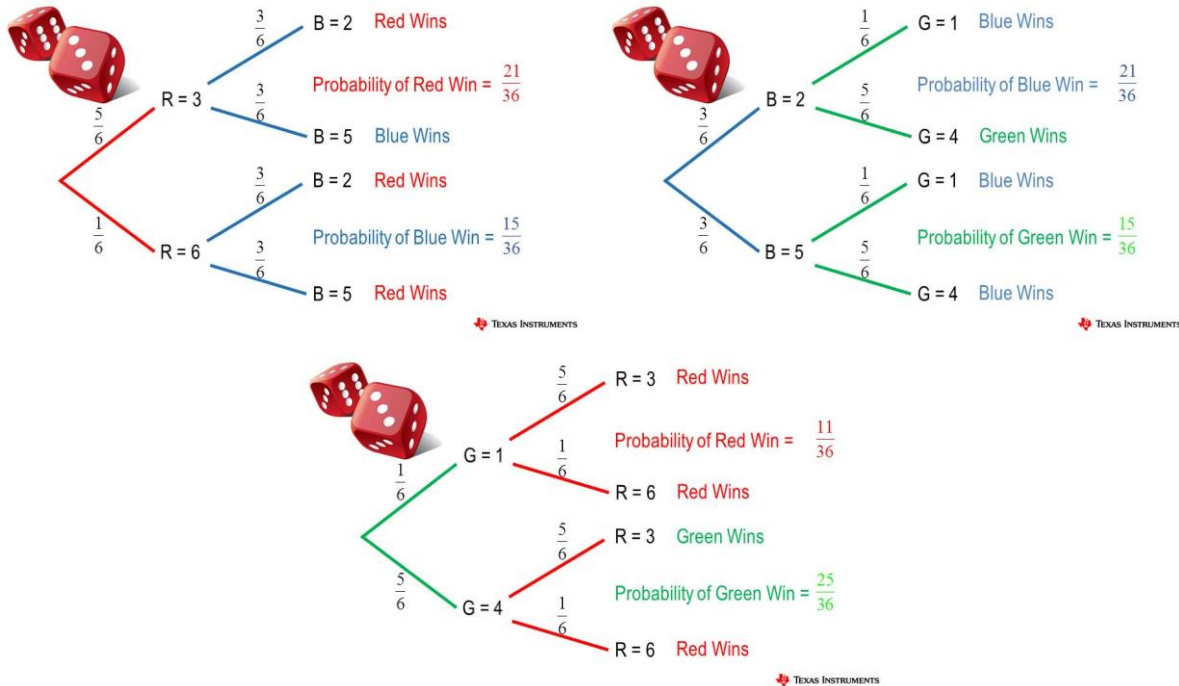
**Question: 12.**

Simulate 100 games of Green vs Red and record the overall results. Discuss the outcome.

Green wins almost every series of 100 games against Red. So we see that Red beats Blue, Blue beats Green and Green beats Red! (Rock – Paper – Scissors = Non-Transitive)

**Question: 13.**

Draw probability tree diagrams for each game: Red vs Blue, Blue vs Green and Green vs Red. Use your tree diagrams to explain the results you obtained for Questions 5, 6 and 8.



Red beats Blue, Blue beats Green and Green beats Red. These results confirm the non-transitive property established using the experimental simulations. Playing this game is therefore unfair for whoever chooses their dice first. If your opponent chooses the Red die, then you should choose Green. If your opponent chooses the Green die, then you should choose the Blue; finally, if your opponent chooses the Blue you should choose Red.

**Question: 14.**

Explain why the 'expected' value does not help determine which dice is likely to win.

The size of the win is irrelevant. A single large number on just one side of the die will increase the expected value but provide no further likelihood of a win. The expected value for the three Grime dice has been manipulated in such a way that each one has the same expected value, a lovely illustration that the expected value is not indicative of the likelihood of a win.