Exploration 26

Teacher Notes

Exploration: Solving a Pair of Linear Equations by Graphing

Learning outcomes addressed

5.1 Graph a pair of linear equations and estimate their solution using the *Intersection Point(s)* command.

5.6 Verify the solution of a linear system by substitution of the solution into both equations.

Lesson Context

The term "intersection" is familiar to all students as it applies to the intersection of two streets. The intersection of the I-95 and 407 Highways in Florida, provides a concrete model for the intersection of two lines. Both highways are close approximations to straight lines near their point of intersection, so we are able to represent them by linear equations in a coordinate system.

In this context, we can talk about the equation of a I-95 as the relationship that is satisfied only by points on highway I-95. Similarly the equation of Highway 407 is the mathematical equation satisfied only by points (or locations) on Highway 407. The intersection of these two highways is the only point that lies on both highways and therefore, satisfies both equations.

Lesson Launch

Have students read the copy beside the map. Ask initiating questions such as:

- What is the relationship between the first and second coordinates of a location on I-95? on Highway 407?

  answer: Each equation defines the relationship.
- How many points or locations are on both highways?
- Is there a location that is on both highways but is not at their intersection? How do you know?

  answer: No. If a point lies on both highways, it must be a common point, i.e. an intersection. Non-parallel straight lines have only one point of intersection.

Lesson Closure

Review with the students the meaning of the equation of a line as the condition that a point lie on the line. Use this idea to reinforce the understanding that the solution of a system of two or more equations is the ordered pair (x, y), or ordered pairs that satisfy all the equations in that system. Mention that most systems of equations in real-world applications involve many equations and must be solved using technology.

Student Work Sheet

Exploration: Solving a Pair of Linear Equations by Graphing

Read the discussion about the intersecting highways in Florida near the Space Center.

❶. Complete each statement.

a) The equation of Highway 407 is $f1(x)$ = ________________________________.

b) The equation of Highway I-95 is $f2(x)$ = ________________________________.

❷. To access the *Graphs* application of TI-*n*spire, press (on) ▸ (enter).

Define $f1(x)$ and $f2(x)$ from ❶ using the entry line at the bottom of the screen.

(You can access the entry line by pressing (tab).)

❸. To find the point of intersection of the lines defined by $f1(x)$ and $f2(x)$, press:

(menu)(7)(3)

Using the pointer tool, click on each line in turn to select it.

The coordinates of the point of intersection of the two lines are: (_____ , ______).

❹. A manufacturer of basketball shoes offers two brands: the light-weight *floaters* @ \$89.50 and the *superstars* @ \$123.99. A basketball team ordered 10 pairs at a total cost of \$998.47. If x denotes the number of *floaters* and y the number of *superstars* ordered, write an equation for the total cost of the 10 pairs.

__________________________________ = \$998.47 ①

Write an equation stating that the total number of pairs of shoes is 10.

_______________ = 10 ②

❺. Solve equation ① for y to express y as an expression in x. Graph this function ① .

To construct a function table for y, press (menu)(2)(9).

Scroll down your function table until you find a pair of integral values of x and y.

The solution of equations ① and ② is x = ________ , y = __________

❻. Verify your solution in ❺ by substituting its coordinates into equations ① and ②.

TI-nspire Investigation

Follow the instructions in the TI-*n*spire Investigation in the *Exercises*.
Then complete these statements:

a) $f1(x)$ = ________________________ b) $f2(x)$ = ________________________

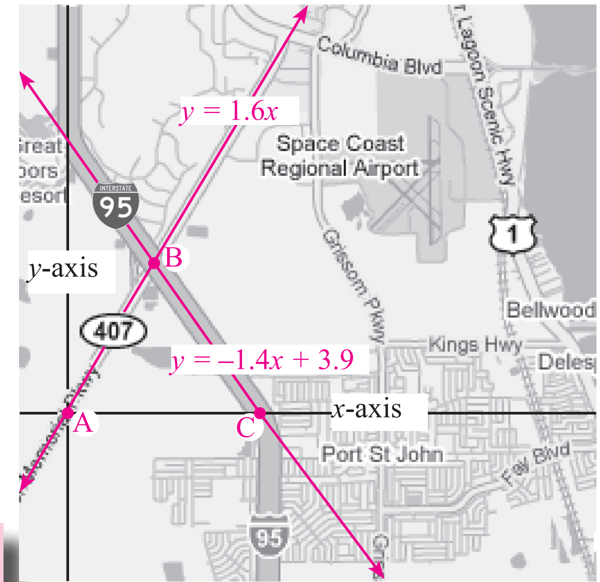
c) The point of intersection of $y = f1(x)$ and $f2(x)$ is (_______, ________).

d) The area of A is __________ . e) d) The area of B is __________ .

Exploration: Solving a Pair of Linear Equations by Graphing

America is Connected by a Network of Highways

The towns, villages and cities of America are connected by a network of highways. Every winter, millions of Americans use highways like I-95 to migrate to states in the sun belt. Determining where these highways will intersect is an important part of planning their routes. The map shows that I-95 runs northwest from point C near Port St. John intersecting Highway 407 at point B. If point A is taken as the origin of a coordinate system with axes as shown, then highways I-95 and 407 are represented by lines with equations $y = -1.4x + 3.9$ and $y = 1.6x$ respectively (where x and y are measured in kilometers). *Example 1* shows how we can use these equations to find the coordinates of point B.

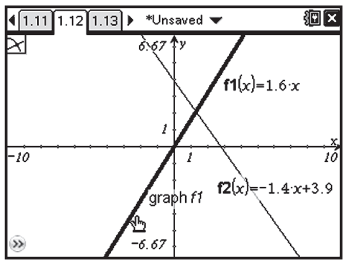


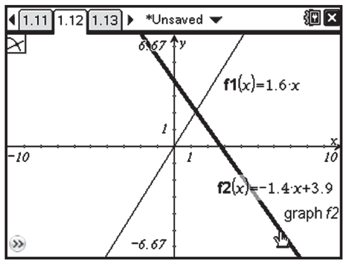
Example 1

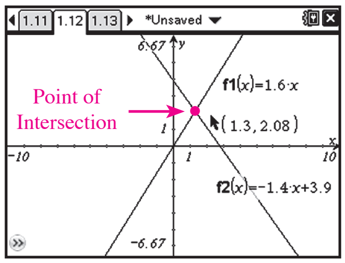
Use the equations of I-95 and Highway 407 to determine the coordinates of point B at the intersection of these highways.

Solution

To access the *Graphs* application, we press: (on) ▸ (enter) and we define $f1(x) = 1.6x$ in the entry line and press (enter). The graph of this equation appears in the display. Then we define $f2(x) = -1.4x + 3.9$ in the entry line and press (enter) to obtain the graph of this equation, as highlighted in the middle display.







To determine the point at which these two graphs intersect, we enter:

(menu) > **Points & Lines** > **Intersection Point(s)** [or (menu) (7) (3)].

Then we move the pointer toward either line until it pulsates and click on that line (i.e., press (enter)) to select it. We then use the pointer to select the other line. The coordinates (1.3, 2.08) appear on the screen at the point of intersection.

Since the point (1.3, 2.08) lies on Highway 407, it satisfies the equation:

$$y = 1.6x \qquad ❶$$

Also (1.3, 2.08) satisfies the equation: $y = -1.4x + 3.9$ ❷

Since the point of intersection is the only point that lies on both lines, it is the only point whose coordinates satisfy both equations.

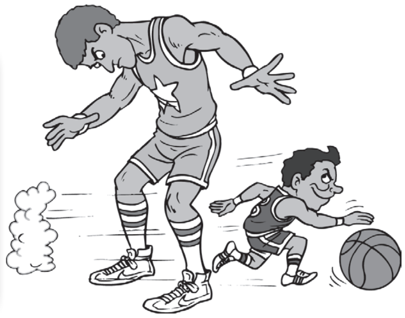
We can verify this by substituting $x = 1.3$, $y = 2.08$ into both equations.

Example 1 introduces us to problems in which we seek an ordered pair that satisfies two linear equations in two variables. In general:

Definition: Two or more linear equations in the same variables is called a *system of linear equations*. The *solution* of this linear system is the ordered pair (x, y) that satisfies all the equations.

Worked Examples

Example 2



A manufacturer of basketball shoes offers two brands: the light-weight *floaters* @ \$89.50 and the *superstars* @ \$123.99. A basketball team ordered 10 pairs at a total cost of \$998.47. Determine how many pairs of each type were ordered:

a) by constructing a table of values.
b) by graphing.

Solution

a) We let x and y represent respectively the number of pairs of *floaters* and *superstars* purchased.

Since there were 10 pairs of shoes purchased, we write: $x + y = 10$. ❶
The total cost of the shoes in dollars is 998.47, i.e., $89.50x + 123.99y = 998.47$. ❷

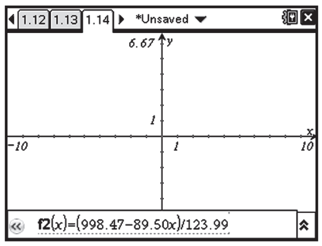
To solve this pair of equations, we must find the ordered pair (x, y) that satisfies both equations. To create a table of values for y corresponding to various values of x, we put equation ❷ in slope-intercept form by solving for y.

Subtracting $89.50x$ from both sides of equation ❷ yields:

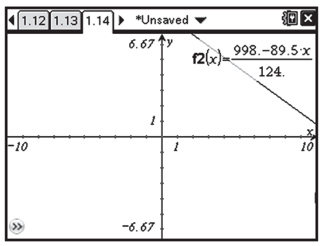
$$123.99y = 998.47 - 89.50x. \quad ❸$$

Dividing both sides of equation ❸ by 123.99 yields $y = \dfrac{(998.47 - 89.50x)}{123.99}$.

To access *Graphs*, we press: (on) ▸ (enter), and we define $f2(x)$ in the entry line as shown in the top display.

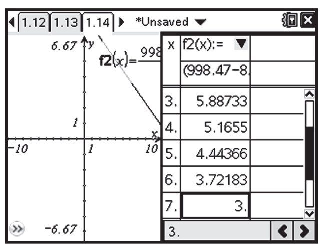


On pressing (enter), we obtain the graph shown in the middle display.



To construct a table of values for the function $f2(x)$, we press:

(menu) > **View** > **Show Table** **[**or (menu)(2)(9)] and obtain the bottom display.



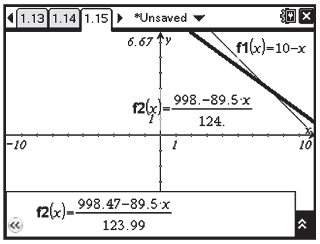
Since x and y must be whole numbers, we use the NavPad, to scroll down to an ordered pair of integers. This display shows that (7, 3) is an ordered pair of integers that satisfies equation ❷. Since 7 + 3 =10, this pair also satisfies equation ❶. Therefore, (7, 3) is a solution to this pair of equations.

That is, $x = 7$, $y = 3$, so the team ordered 7 pairs of *floaters* and 3 pairs of *superstars.*

We observe that this method of solving two equations in two variables can be applied when the solution is a pair of integers. However, in most cases, a more general method is required, such as the solution by graphing shown in *part b* (on the next page).

Solution

b) To create a full screen in the *Graphs* application, we press: (on) ▸ (enter), and recall $f2(x)$ in the entry line. The graph of equation ❷ above appears.



Why does the fraction opposite the graph have denominator 124, but the fraction in the entry line has denominator 123.99?

To graph equation ❶, we solve for y, subtracting x from both sides of the equation to obtain $y = 10 - x$. We enter $f1(x) = 10 - x$ into the entry line and press (enter). We press:

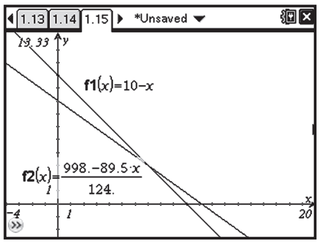
(menu) > **View** > **Hide Entry Line** to hide the entry line.

All the points that satisfy equation ❶ lie on the graph of $f1(x)$.
All the points that satisfy equation ❷ lie on the graph of $f2(x)$.
To satisfy both equations, a point must lie on both graphs, that is, on their point(s) of intersection.

To find the point of intersection, we must first choose an appropriate window for displaying the area of the graphs where they intersect. Since the intersection point is in the first quadrant, we press:

(menu) > **Window / Zoom** > **Quadrant 1**

and we obtain the middle display.

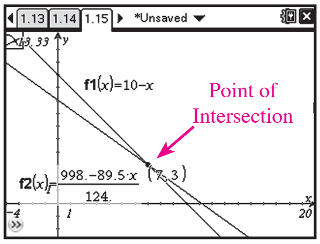


To find the intersection point, we press:

(menu) > **Points & Lines** > **Intersection Point(s)**

and move the pointer toward either line until it pulsates. Then click on that line (i.e., press (enter)) to select it, and use the pointer to select the other line.

The coordinates (7, 3) appear on the screen at the point of intersection, verifying the solution in *part a*.



Example 3 shows how we can verify that our solution satisfies the linear system.

Example 3

Verify that (7, 3) is the solution to the linear system:

$x + y = 10.$ ❶

$89.50x + 123.99y = 998.47.$ ❷

Solution

To prove that (7, 3) is the solution to the given linear system, we must show that (7, 3) satisfies both equations. When we substitute $x = 7$, $y = 3$ into equation ❶ we obtain :
Left side = 7 + 3; Right side = 10. Since the left and right sides are equal ❶ is satisfied.

When we substitute $x = 7$, $y = 3$ into equation ❷ we obtain :
Left side = 89.5(7) + 123.99(3) or 998.47 ; Right side = 998.47. Since the left and right sides are equal, ❷ is satisfied. Since (7, 3) satisfies both equations, it is the solution to the system.

Exercises and Investigations

❶. Here is a list of ordered pairs: {(7,2), (8,1), (2,7), (5,1)}
Two equations are: $x + y = 9$ ①
and $2x + y = 11$ ②
a) List the ordered pairs that satisfy equation ①.
b) List the ordered pairs that satisfy equation ②.
c) List the ordered pairs that satisfy equations ① and ②.
d) What is the solution to both equations ① and ②?

❷. How many solutions has a pair of linear equations? Explain how you can determine the number of solutions by examining the graphs of the two equations.

❸. Isolate y in each equation and graph the expression as a function of x.

$$x + y = 1$$
$$3x - y = 11$$

Press [menu] [7] [3] and select each line. Record the point of intersection. Verify that your solution satisfies both equations.

❹. Isolate y in each equation and graph the expression as a function of x.

$$4x + y = -5$$
$$2x + 3y = 5$$

Press [menu] [7] [3] and select each line. Record the point of intersection. Verify that your solution satisfies both equations.

❺. Two poles of height 6 m and 24 m are 30 m apart. How high above the ground is the point of intersection of the lines running from the top of each pole to the foot of the other?

❻. The cost C (in dollars) of a monthly membership at the Kuts 'n Kurves Health Club is a fixed monthly fee plus a fixed towel charge per visit. The table shows the amounts that Alexis paid and her number of visits to the Club in April and May.

Month	April	May
Visits	15	12
Cost	$66.50	$56.00

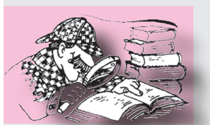
Let x denote the number of visits and y the fixed monthly fee.
a) Write an equation in x and y stating that the total cost for 15 visits was $66.50. Write this equation in the form $y = f(x)$. Define it in the *Graphs* application as $f1(x)$.
b) Write an equation in x and y stating that the total cost for 12 visits was $56.00. Write this equation in the form $y = f(x)$. Define it in the *Graphs* application as $f2(x)$.
c) Graph $f1(x)$ and $f2(x)$ and select an appropriate window by pressing [menu] [4] [A].
d) Find the point of intersection of the two graphs by pressing [menu] [4] [A] and selecting each line as in *Example 1*.
e) What is the fixed monthly fee and the cost per visit?
f) What would it cost Alexis for a month with 20 visits?

❼. To print a book, the Acme Printing House charges a press setup fee plus a fixed amount per book. The table shows the total costs to a publisher to print various numbers of books.

The Cost of Printing Books		
Number	7500	10,000
Cost	$40,575.00	$53,700.00

Let x denote the fixed amount per book and y denote the press setup fee.
a) Write an equation in x and y stating that the cost of printing 7500 books is $40,575. Write this equation in the form $y = f(x)$. Define it in the *Graphs* application as $f1(x)$.
b) Write an equation in x and y stating that the cost of printing 10,000 books is $53,700. Write this equation in the form $y = f(x)$. Define it in the *Graphs* application as $f2(x)$.
c) Graph $f1(x)$ and $f2(x)$ and select an appropriate window by pressing [menu] [4] [A].
d) Find the point of intersection of the two graphs by pressing [menu] [7] [3] and selecting each line as in *Example 1*.
e) What is the press set up fee and the cost per book?

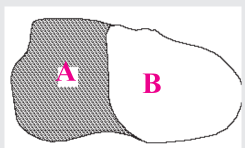
TI-nspire Investigation



A Problem from Four Thousand Years Ago

The problem below was found (without metric units) on a clay tablet dating back to the ancient Babylonian civilization. It was particularly difficult for the students in those ancient times because their TI-*n*spire calculators had clay keys.

A plot of land with an area of 1800 m² is composed of two parts. The more fertile part A yields about 2/3 kg of wheat per square meter. The less fertile part B yields about 1/2 kg of wheat per square meter. In total, A yields 500 kg more wheat than B. What is the area of each part?



Let x and y denote respectively the areas of A and B in m^2.
a) Write an equation in x and y that states that the total area is 1800 m^2. Write this equation in the form $y = f(x)$. Define it in the *Graphs* application as $f1(x)$.
b) Write an equation in x and y that states that A yields 500 kg more wheat than B. Write this equation in the form $y = f(x)$. Define it in the *Graphs* application as $f2(x)$.
c) Graph $f1(x)$ and $f2(x)$ and find the point of intersection of the two graphs as in the previous exercises.
d) What are the areas of A and B?
e) Write the equation $f1(x) = f2(x)$. Solve it for x. Compare with your answer in *part d*.

Answers to the Exercises & Hints for the Investigations

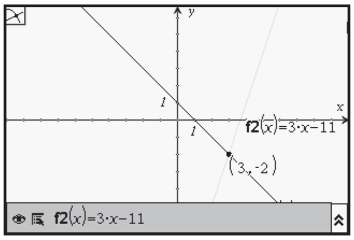
Exploration 26

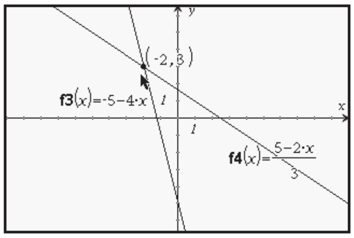
❶. a) (7, 2), (8, 1), (2,7) b) (2, 7), (5, 1)
c) (2, 7) d) (2, 7)

❷. If the two linear equations represent the same line, they have an infinite number of solutions. If they represent two different lines, then they have one solution if they are non-parallel and no solutions if they are parallel.

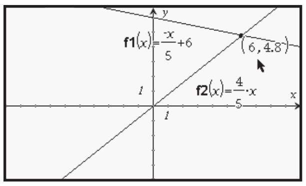
❸. The point of intersection is shown in the display below left.

❹. The point of intersection is shown in the display below right.





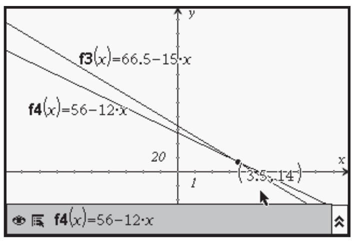
❺. If we place the bottom of the 6-m pole at the origin, the line from the top of that pole to the bottom 24-m pole rises 6 m over a horizontal distance of 30 m. Therefore it has slope 6/30 or 1/5. The line through the top of the 6-m pole to the bottom of the 24-m pole is described by the equation $y = -(1/5)x + 6$. Similarly, the line from the top of the 24-m pole to the bottom of the 6-m pole has slope 24/30 or 4/5 and passes through (0, 0). Its equation is $y = 4/5x$. The point of intersection is (6, 4.8) indicating that the intersection occurs at 4.8 m above the ground.



❻. a) $15x + y = 66.50$
$y = 66.50 - 15x$

b) $12x + y = 56.00$
$y = 56.00 - 12x$

c), d) & e) The display shows that the intersection of the graphs occurs at (3.5, 14). That is, the towel charge per visit is \$3.50 and the fixed monthly fee is \$14.

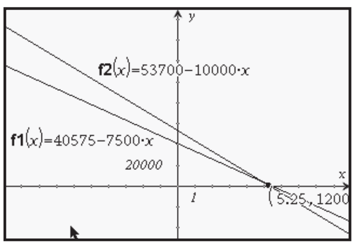


f) The cost of 20 visits would be \$14 + 20(\$3.50) or \$84.

❼. a) $7500x + y = 40{,}575$
$y = 40575 - 7500x$

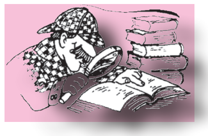
b) $10{,}000x + y = 53{,}700$
$y = 53700 - 10000x$

c), d) & e) The display shows that the intersection of the graphs occurs at (5.25, 1200). That is, the cost per book is \$5.25 and the press set up fee is \$1200.



Exploration 26 *cont'd*

Hint for the TI-nspire Investigation



The equations are:
$x + y = 1800$ and $2/3x - 1/2y = 1800$
The point of intersection is (1200, 600).
The areas of A and B are respectively 1200 m^2 and 600 m^2.