Math Objectives

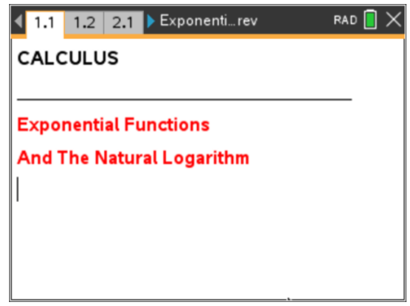
- Students will understand the distinction between growth rate (slope) and relative growth rate (slope/y-value) of a function.
- Students will understand that the relative growth rate $\mathbf{f}'(x)/\mathbf{f}(x)$ is constant for exponential functions of the form $\mathbf{f}(x) = b^x$.
- Students will identify the natural logarithm of the base of an exponential function as this constant relative growth rate.
- Students will describe the behavior of the graph of $y = \mathbf{f}(x) = b^x$ for different values of b.
- Students will understand what is special about the number e as a base for exponential functions in terms of growth rate.
- Students will look for and make use of structure. (CCSS Mathematical Practice)
- Students will reason abstractly and quantitatively. (CCSS Mathematical Practice)

Vocabulary

- relative growth rate
- exponential function
- natural logarithm
- natural exponential function and its base *e*

About the Lesson

- The purpose of this lesson is to guide students to discover a surprising property involving the relative growth rate of an exponential function. Students will consider the ratio $\mathbf{f}'(x)/\mathbf{f}(x)$ for $\mathbf{f}(x) = b^x$ and see that for a given value b this ratio is a constant for all values of x. We define this ratio as the natural logarithm of b, denoted $\ln(b)$.
- Students are asked to explain why the natural logarithm of b is also just the slope of the function $\mathbf{f}(x) = b^x$ at $x = 0$.
- Students will construct a table of values that suggests there exists a value b such that $\ln(b) = 1$. Students are asked to estimate the value of this special value e to three decimal places.
- The activity concludes by revisiting the idea of relative growth rate and the very special property of the natural exponential function being its own derivative: if $\mathbf{f}(x) = e^x$, then $\mathbf{f}'(x) = e^x$.



TI-Nspire™ Technology Skills:

- Download a TI-Nspire document
- Open a document
- Move between pages
- Grab and drag a point

Tech Tips:

- Make sure the font size on your TI-Nspire handheld is set to Medium.
- You can hide the function entry line by pressing ctrl G.

Lesson Materials:

Student Activity

- Exponential_Functions_and_the_Natural_Logarithm_Student.pdf
- Exponential_Functions_and_the_Natural_Logarithm_Student.doc

TI-Nspire document

- Exponential_Functions_and_the_Natural_Logarithm.tns

Visit www.mathnspired.com for lesson updates and tech tip videos.

TI-Nspire™ Navigator™ System

- Use Quick Poll questions and Screen Capture to adjust the pace of the lesson according to student understanding.

Discussion Points and Possible Answers

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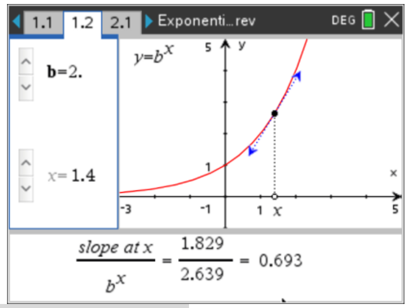

1. For $b = 2$, use the up/down arrows at the lower left of the screen to move the point x along the axis. At the bottom of the page, observe the slope at x, the value of b^x, and the ratio $\frac{\text{slope at } x}{b^x}$.

Teacher Tip: The purpose of this question is for students to discover that the relative growth rate for this exponential function is the same for every value of x.

a. What happens to the slope at x and the value of b^x when you move x to the right of 1?

Answer: Both the slope and the value increase (and are always positive).

b. What happens to the ratio when you move x to the right of 1?

Answer: The ratio remains the same, always 0.693.

c. What happens to the slope at x and the value of b^x when you move x to the left of 1?

Answer: Both the slope and the value decrease (but remain positive) as you move x to the left.

d. What happens to the ratio when you move x to the left of 1?

Answer: The ratio still remains the same, always 0.693.

Teacher Tip: Many Algebra and Precalculus texts introduce properties of logarithms using the natural logarithm of 2, 3, 4,…. Therefore, it is not unusual for students to remember and recognize 0.693 as ln(2).

2. Use the up/down arrows at the upper left to change the base of the exponential function. Set $b = 3$.
   a. Move the value x along the axis and observe the ratio. Describe what happens to the relative growth rate as x varies.

Answer: The relative growth rate remains the same, 1.099.

   b. Try other values for b. What happens to the relative growth rate as x varies, for any set value of b?

Answer: For any value of b, the relative growth rate is constant. For example, for $b = 4$, the relative growth rate is equal to 1.386, and for $b = 0.5$, the relative growth rate is equal to –0.693.

   c. As b increases, what happens to the relative growth rate of the exponential function?

Answer: As b increases, the relative growth rate also increases.

   d. Is the relative growth rate ever 0? If so, for what value(s) of b?

Answer: The relative growth rate is 0 for $b = 1$. In this case, the exponential function is $\mathbf{f}(x) = 1^x = 1$ and the graph of $y = \mathbf{f}(x)$ is a horizontal line, with constant slope 0.

   e. Is the relative growth rate ever negative? If so, for what value(s) of b?

Answer: The relative growth rate is negative for $0 < b < 1$.

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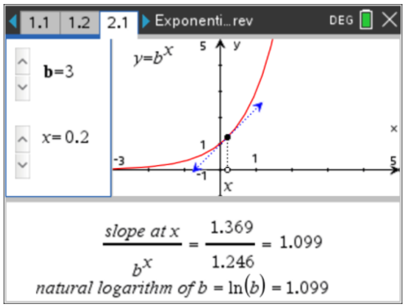


3. For any value of b, the constant relative growth rate of the exponential function $\mathbf{f}(x) = b^x$ is called the natural logarithm of b, abbreviated $\ln(b)$. Use the arrow keys to change the value of b and complete the following table of corresponding values for $\ln(b)$.

Teacher Tip: The relationship between the natural logarithm of a number and the natural logarithm of its reciprocal is evident from the table.
Questions that could be posed here are:
What do you think ln(10) will be?
How about $\ln\left(\frac{1}{3}\right)$?
What is the general relationship between ln(*a*) and $\ln\left(\frac{1}{a}\right)$?

b	0.1	0.2	0.4	0.5	1	2	2.5	3	5
ln(*b*)	–2.303	–1.609	–0.916	–0.693	0	0.693	0.916	1.099	1.609

a. Explain what happens to the graph of $y = \mathbf{f}(x) = b^x$ as *b* increases and is greater than 1.

Answer: The graph is always increasing, and becomes steeper and steeper for positive *x* as *b* increases. (Some students may further comment that the graph is always concave up.)

b. Explain what happens to the graph of $y = \mathbf{f}(x) = b^x$ as *b* becomes less than 1.

Answer: The graph is always decreasing, and becomes steeper and steeper for negative *x* as *b* decreases. (Some students may comment that the graph is still always concave up.)

c. Explain the value of ln(1) geometrically.

Answer: When *b* = 1, the function is the constant function **f**(*x*) = 1 for all *x*. Its graph is a horizontal line, and thus the slope and relative growth rate are 0.

d. A student claims that you can also find ln(*b*), the natural logarithm of *b*, by just looking at the slope of the graph of $y = \mathbf{f}(x) = b^x$ at *x* = 0. Is this correct? Why or why not?

Answer: Yes, this is correct. Since the relative growth rate is constant for all *x*, we can determine it at any value *x* we choose. At *x* = 0, the denominator of the ratio is $b^0 = 1$, so the slope value in the numerator gives us the natural logarithm of *b* immediately.

4. Find the following natural logarithms using the up/down arrows for these values of *b*.
   ln(2.6) = ________________ ln(2.7) = __________________ ln(2.8) = __________________

   There is a value *b* such that ln(*b*) = 1. The exact value of this special number is labeled *e*.

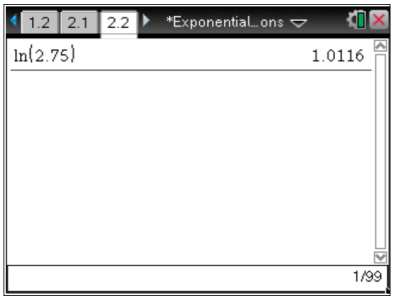
   <u>Answer:</u> ln(2.6) ≈ 0.956; ln(2.7) ≈ 0.993; ln(2.8) ≈ 1.030

TI-Nspire Navigator Opportunity: *Live Presenter*
See Note 1 at the end of this lesson.

5. The value of *e* lies between two of these values of *b:* 2.6, 2.7, and 2.8. Which two? Explain your reasoning.

   <u>Answer:</u> It lies between 2.7 and 2.8 (or 2.6 and 2.8). Most students will reason that since 1 is between the natural logarithm values of 2.7 and 2.8, then *e* must be between 2.7 and 2.8.

> **Teacher Tip:** The reasoning given above is correct, but it is based on assumptions about the behavior of ln(*b*) as *b* changes that need to be noted. In particular, does ln(*b*) strictly increase as *b* increases? (Yes, but why does that make sense?)



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6. Is *e* smaller or larger than 2.75? Explain your reasoning.

   <u>Answer:</u> If the values of ln(*b*) strictly increase as *b* increases, then 2.75 must be larger than *e*, since ln(2.75) = 1.012 > 1.

7. Use this Calculator page to evaluate the natural logarithm in search of the closest three-decimal-place approximation to *e* you can make.
   a. What is your (three-decimal-place) approximation?

   <u>Answer:</u> *e* ≈ 2.718

> **Teacher Tip:** The strictly increasing nature of ln(*b*) is critical to use here in "homing in" on a three-decimal-place approximation of *b*.

b. What is the base of the natural exponential function $\mathbf{f}(x) = b^x$?

Answer: The special number, *e*, must also be the base of the natural exponential because we know that the natural logarithm of the base gives the relative growth rate and ln(*e*) = 1.

c. What is its derivative $\mathbf{f}'(x)$?

Answer: The derivative of $\mathbf{f}(x) = e^x$ must be itself. $\mathbf{f}'(x) = e^x$. For $b = e$, $\frac{\mathbf{f}'(x)}{\mathbf{f}(x)} = \ln(e) = 1$,

and therefore, $\mathbf{f}(x) = \mathbf{f}'(x) = e^x$. This is the only base for which this is true. In words, for $b = e$, the slope of the exponential function at any point $x = a$ is always the same as the value of the function at $x = a$.

TI-Nspire Navigator Opportunity: *Quick Poll*
See Note 2 at the end of this lesson.

Wrap Up

This activity leads students through one derivation of the natural logarithm and its relationship to exponential functions. Upon completion of the discussion, the teacher should ensure that students understand:

- The definition of relative growth rate as the ratio of $\mathbf{f}'(x)$ to $\mathbf{f}(x)$.
- That the relative growth rate for any exponential function $\mathbf{f}(x) = b^x$ is constant, and this constant defines ln(*b*), the natural logarithm of the base *b*.
- That the relative growth rate for an exponential function $\mathbf{f}(x) = b^x$ is also the slope of the exponential function at $x = 0$, namely ln(*b*).
- The graphical behavior of functions in the form $\mathbf{f}(x) = b^x$ for:
  - $0 < b < 1$: graph is always positive (above *y*-axis), strictly decreasing, always concave up
  - $b = 1$: a constant function with the horizontal line $y = 1$ as its graph
  - $b > 1$: graph is always positive (above *y*-axis), strictly decreasing, always concave up
- The definitions of the special number *e* and the natural exponential function.
- Why the derivative of the natural exponential function is itself.

The number *e* is a special character found using the e^x key on the calculator (not the alphabetic "e" on the keyboard). Press e^x to evaluate $e^1 = 1$ or to graph the function e^x. The notation exp(*x*) is an alternative notation for the natural exponential recognized by the calculator, and it can also be directly typed using the keyboard.

Here is a good follow-up TI-Nspire exercise to this activity:

1. Open a new *Graphs* application in a new page.
2. Graph the natural exponential function **f1**(x) = e^x.
3. Use **Menu > Points & Lines > Tangent** to put a tangent line on the graph.
4. Use **Menu > Actions > Coordinates and Equations** to put the coordinates on the point of tangency.
5. Use **Menu > Measurement > Slope** to measure the slope of the tangent line.
6. Move the point of tangency to see that the *y*-coordinate of the point and the slope always match.

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Note 1

Question 4, *Live Presenter*

Have one student act as the Live Presenter, using the up/down arrows of the slider to find the value of ln(2.6). Have two other students find the other two values. Encourage other students in the classroom to help the Live Presenter find the correct value.

Note 2

Question 7 part c, *Quick Poll*

Send a Quick Poll to students to see how many are able to determine the derivative of e^x. Discuss what may have led to the incorrect responses.